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CYCLIC COHOMOLOGY, THE NOVIKOV CONJECTURE AND HYPERBOLIC GROUPS

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INTRODUCTION

NOVIKOV's conjecture on the homotopy invariance of higher signatures [28] can be formulated as follows: given a finitely presented group Γ and a compact oriented smooth manifold M, together with a continuous map $\psi: M \to B\Gamma$, the generalized signatures $\langle L(M)\cdot\psi^*(\xi), [M]\rangle$, where ξ runs over all classes in $H^*(B\Gamma, \mathbb{Q})$ and $L(M)$ denotes the total Hirzebruch L-class of M, are homotopy invariants of the pair (M, ψ). In other words, if $h: N \to M$ is a homotopy equivalence of oriented smooth manifolds, then $\langle L(N)\cdot h^*(\psi^*(\xi)), [N]\rangle = \langle L(M)\cdot\psi^*(\xi), [M]\rangle$. The validity of this conjecture has been established, by a variety of techniques, for many groups Γ, most notably for closed discrete subgroups of finitely connected Lie groups. The latter result is due to Kasparov [24] and its proofs is based on bivariant K-theory.

In this paper we present a new and more direct method for attacking the Novikov conjecture, which yields a proof of the conjecture for Gromov's (word) hyperbolic groups [18]. These groups form an extremely rich and interesting class of finitely presented groups, which differs significantly, both in size and in nature, from the groups for which Novikov's conjecture was previously known. First of all, as pointed out by Gromov [18], they are "generic" among all finitely presented groups in the following sense: the ratio between the number of hyperbolic groups and all groups with a fixed number of generators and a fixed number of relations, each of length at most l, tends to 1 when $l \to \infty$ [18, 0.2(A)]. Secondly, when adding at random relations to a (non-elementary) hyperbolic group, one obtains again a hyperbolic group [18, 5.5]. Thirdly, the cohomology of any finite polyhedron can be embedded into the cohomology of a hyperbolic group [18, 0.2(c)]. Also, many of the hyperbolic groups exhibit "exotic" properties, like Kazhdan's property T [18, 5.6] or being non-linear (in a non-trivial way).

Our approach is based on expressing the higher signatures in terms of the pairing between cyclic cohomology and K-theory (cf. [8]). The hyperbolicity assumption plays a twofold role: first, it ensures, via a deep result of Gromov [18; 8.3T], that every class $\xi \in H^{k\geq 2}(B\Gamma, \mathbb{C})$ can be represented by a bounded group cocycle, and secondly, it enables us to make use of a critical norm estimate (first proved by Haagerup [20] for free groups), recently extended by Jolissaint [23] and de la Harpe [21] to hyperbolic groups.

The paper is organized as follows. Using the Alexander–Spanier realization of the cohomology of a smooth manifold, reviewed in §1, we define in §2 *localized analytic indices*

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for an elliptic operator. Then, in §3 we prove a refinement of the Atiyah–Singer Index Theorem, giving a cohomological formula for these "higher" indices. In §5, we extend this theorem to covering spaces, which provides us with a powerful tool for attacking the Novikov conjecture. An alternate route to the Higher Index Theorem for covering spaces is sketched in a remark at the end of §5. As a by-product of the proof of the Localized Index Theorem, we were led to construct a cohomology theory of de Rham type for arbitrary C^*-algebras (to be discussed in another paper). This is touched upon in §4, where we digress to explain the link between localized indices and entire cyclic cohomology [9]. Finally, §6 contains the proof of the Novikov conjecture for hyperbolic groups.

The main results of this paper have been announced in [12]. We have reasons to believe that by using entire cyclic cohomology rather than standard (polynomial) cyclic cohomology, the scope of our method can be enlarged to encompass the groups which Gromov calls semihyperbolic [18, 0.2(E), Non-definition].

§1. ALEXANDER–SPANIER COHOMOLOGY

The Alexander–Spanier version of the cohomology of a smooth manifold will be an important ingredient in our construction of higher indices. For the convenience of the reader unfamiliar with it, we devote this preliminary section to a review of some aspects of it which are needed in the present paper.

Let us start recalling the definition of Alexander–Spanier cohomology with real coefficients, of a topological space M (for details, see [32; Chap. 6]). With $q \geq 0$, let $C^q(M)$ be the vector space of all functions φ from M^{q+1} to $\mathbb{R}$; a coboundary homomorphism δ: $C^q(M) \to C^{q+1}(M)$ is defined by the formula

$$(\delta\varphi)(x^0, \ldots, x^{q+1}) = \sum_{i=0}^{q+1} (-1)^i \varphi(x^0, \ldots, x^{i-1}, x^{i+1}, \ldots, x^{q+1}),$$

and $C^*(M) = \{C^q(M), \delta\}$ is a cochain complex over $\mathbb{R}$. Its cohomology is trivial, except in dimension 0. The nontrivial cohomological information is concentrated in the subcomplex of "locally zero" cochains. An element $\varphi \in C^q(M)$ is said to be *locally zero* if there is an open covering $\mathscr{U}$ of M such that φ vanishes on the neighborhood $\mathscr{U}^{q+1} = \bigcup_{U \in \mathscr{U}} U^{q+1}$ of the qth diagonal of M. If φ is locally zero then, evidently, so is $\delta\varphi$. One obtains thus a subcomplex $C_0^*(M) = \{C_0^q(M), \delta\}$ of $C^*(M)$. The corresponding quotient complex $\bar{C}^*(M) = \{\bar{C}^q(M), \delta\}$ is called the *Alexander–Spanier complex* of M with coefficients in $\mathbb{R}$ and its graded cohomology space $\bar{H}^*(M)$ is called the *Alexander–Spanier cohomology* of M (with real coefficients). If $\varphi \in C^q(M)$, we shall denote by $\bar{\varphi}$ its image in $\bar{C}^q(M)$ and by $[\bar{\varphi}]$ the corresponding cohomology class.

From now on we shall assume that M is an m-dimensional, oriented, C^∞ manifold (Hausdorff and with a countable basis of open sets). A more appropriate Alexander–Spanier complex for this situation is $\bar{C}_\infty^*(M) = C_\infty^*(M)/C_{\infty,0}^*(M)$ defined in the obvious way by means of C^∞ cochains. At some point we shall also need the Borelian version, $\bar{C}_\beta^*(M) = C_\beta^*(M)/C_{\beta,0}^*(M)$, defined in terms of Borel cochains. We let $\bar{H}_\infty^*(M)$, resp. $\bar{H}_\beta^*(M)$ denote the corresponding graded cohomology spaces.

(1.1) LEMMA. *The canonical homomorphism from $\bar{H}_\infty^*(M)$, resp. $H_\beta^*(M)$, to $\bar{H}^*(M)$ is an isomorphism.*

Proof. The standard proof of the fact that $\bar{H}^*(M)$ is naturally isomorphic to the Čech cohomology $H^*(M)$ with coefficients in the constant sheaf $\mathbb{R}$ [32; Sec. 6.8], applies as well to $\bar{H}^*_\infty(M)$, resp. $\bar{H}^*_\beta(M)$. □

In view of the above lemma, we shall suppress the subscript ∞ (resp. β) in the notation of the Alexander–Spanier cohomology. In fact, since we shall always be dealing with C^∞ manifolds, we shall also omit from now on the subscript ∞ in the notation of the Alexander–Spanier complex.

In order to define a homomorphism of complexes from $\bar{C}^*(M)$ to the de Rham complex $\Lambda^*(M)$, we endow M with a Riemannian metric and choose an open covering $\mathscr{B}$ with the following properties:

(a) $\mathscr{B}$ is locally finite;
(b) each $B \in \mathscr{B}$ is a geodesically convex ball, whose center and radius will be denoted x_B and r_B, respectively;
(c) the square of the Riemannian distance, $d^2(x, y)$, is a C^∞ function of x and y on $\mathscr{B}^2$;
(d) if $B \in \mathscr{B}$ and $x^0 \in B$, the function sending $v \in T_{x_B}M$, $\|v\| < r_B$ to $d^2(x^0, \exp_{x_B} v)$ has positive definite Hessian on $\exp_{x_B}^{-1}(B)$;
(e) the sets $B_{1/3} = \{x \in M;\ d(x_B, x) < r_B/3\}$ form a covering $\mathscr{B}_{1/3}$ of M.

All the required properties, except perhaps (d), are standard. Concerning (d), there is in fact a local expansion for the square of the distance in a normal coordinate neighborhood (cf. [13, (2.2)]) which gives the strengthened version:

$$\begin{aligned} d^2(x^0, x) &= \|v - v^0\|^2 - \frac{1}{3} \sum_{i,j,k,l} R_{ijkl}(x_B) v_j^0 v_l^0 (v_i - v_i^0)(v_k - v_k^0) \\ \text{(d}') \qquad & + \text{higher order terms,} \end{aligned}$$

where $x^0 = \exp_{x_B} v^0$, $x = \exp_{x_B}(v)$.

Let Σ^q denote the standard simplex $\{\mathbf{t} = (t_0, \dots, t_q) \in [0, 1]^{q+1};\ t_0 + \dots + t_q = 1\}$ in $\mathbb{R}^{q+1}$.

(1.2) LEMMA. *If* $\mathbf{x} = (x^0, \dots, x^q) \in \mathscr{B}^{q+1}_{1/3}$ *and* $\mathbf{t} = (t_0, \dots, t_q) \in \Sigma^q$, *the function sending* $y \in M$ *to* $\sum_{i=0}^q t_i d^2(x^i, y)$ *has a minimum which is attained in a unique point* $\sum_{i=1}^q t_i x^i \in M$. *Moreover, this point depends differentiably on* $(\mathbf{x}, \mathbf{t}) \in \mathscr{B}^{q+1}_{1/3} \times \Sigma^q$.

Proof. Choose $B \in \mathscr{B}$ such that $\mathbf{x} \in B^{q+1}_{1/3}$. Let μ_B be the minimum value of the above function on $\bar{B}$. Note that

$$\mu_B \le \sum_{i=0}^q t_i d^2(x^i, x^0) \ge r_B^2/9.$$

Hence, if the function takes a smaller value at some $y \in M$, then

$$\min_{0 \le i \le q} d(x^i, y) \le r_B/3.$$

Therefore, $d(x_B, y) \le 2r_B/3$, in particular $y \in B$. This shows that the minimum exists and it is attained in B. Now in B, due to (d'), our function has only one critical point, which, by the implicit function theorem, depends differentiably on $\mathbf{x}$ and $\mathbf{t}$. □

Given $\mathbf{x} \in \mathscr{B}^{q+1}_{1/3}$ we now define a C^∞ simplex $s_q[\mathbf{x}]: \Sigma^q \to M$ by setting

$$s_q[\mathbf{x}](t_0, \dots, t_q) = \sum_{i=0}^q t_i x^i.$$

Together with the covering $\mathscr{B}$, it will be convenient to fix a collection of functions $\chi = \{\chi_q\}_{q \geq 0}$ such that:

(f) $\chi_q \in C^\infty(M^{q+1})$, support $\chi_q \subset \mathscr{B}^{q+1}$ and $\chi_q \equiv 1$ on a neighborhood of the qth diagonal in M^{q+1};

(g) $\chi_q(x^{\tau(0)}, \ldots, x^{\tau(q)}) = \chi_q(x^0, \ldots, x^q)$, $\forall \tau \in \mathscr{S}_{q+1}$ = the permutation-group of order $(q+1)!$.

Let now $\Lambda^*(M) = \{\Lambda^q(M), d\}$ be the de Rham complex of differential forms on M. Given $\omega \in \Lambda^q(M)$, we define $\rho(\omega) \in C^q(M)$ by

$$\rho(\omega)(x^0, \ldots, x^q) = \chi_q(x^0, \ldots, x^q) \int_{s_q[x^0, \ldots, x^q]} \omega.$$

The vanishing of χ_q outside $\mathscr{U}^{q+1}$ gives an obvious meaning to the right hand side for any $(x^0, \ldots, x^q) \in M^{q+1}$. It is also clear that the class

$$\bar{\rho}(\omega) = \overline{\rho(\omega)} \in \bar{C}^q(M)$$

is independent of the choice of $\mathscr{B}$ and χ with the above properties. The map $\bar{\rho}: \Lambda^*(M) \to \bar{C}^*(M)$ thus defined is a homomorphism of complexes, i.e.

$$\delta \overline{\rho(\omega)} = \bar{\rho}(d\omega).$$

Indeed, from the definition of the simplex $s_q[\mathrm{x}]$ it follows easily that its boundary $\partial s_q[\mathrm{x}]$ can be expressed as follows:

$$\partial s_q[x^0, \ldots, x^q] = \sum_{i=0}^{q} (-1)^i s_{q-1}[x^0, \ldots, x^{i-1}, x^{i+1}, \ldots, x^q];$$

thus, the claimed identity is a consequence of Stokes' theorem for chains. □

(1.3) LEMMA. *The induced homomorphism in cohomology $\bar{\rho}^*: H^*_{dR}(M) \to \bar{H}^*(M)$ is an isomorphism.*

Proof. The map ρ induces a homomorphism of presheaves from the de Rham presheaf Λ^* to the Alexander–Spanier presheaf C^*. Since $H^q(\Lambda^*)$ and $H^q(C^*)$ are locally zero (see [32; Chap. 6] for terminology) if $q > 0$, and $H^0(\Lambda^*) \cong H^0(C^*) \cong \mathbb{R}$, the induced homomorphism $\rho^q_*: H^q(\Lambda^*) \to H^q(C^*)$ is evidently a local isomorphism for all $q \geq 0$. From the uniqueness theorem of the cohomology of presheaves [32, Thm. 6.8.9], one obtains, for each $q \geq 0$, an isomorphism

$$\hat{\rho}^q_*: H^q(\hat{\Lambda}^*(M)) \to H^q(\hat{C}^*(M)),$$

where the "roof' signifies passage to the sheaf completion. But Λ^* is already a sheaf, and thus $\hat{\Lambda}^* \cong \Lambda^*$, whereas $\hat{C}^* \cong \bar{C}^*$. □

(1.4) LEMMA. *The following subcomplexes of $\bar{C}^*$ can be used to compute $\bar{H}^*(M)$:*

(i) $\bar{C}^*_\alpha(M) = \{\bar{C}^q_\alpha(M), \delta\}$, $\bar{C}^q_\alpha(M) = C^q_\alpha(M)/C^q_\alpha(M) \cap C^q_0(M)$,

where $C^q_\alpha(M) = \{\varphi \in C^q(M);\ \varphi(x^{\tau(0)}, \ldots, x^{\tau(q)}) = \mathrm{sgn}(\tau)\varphi(x^0, \ldots, x^q), \forall \tau \in \mathscr{S}_{q+1}\}$;

(ii) $\bar{C}^*_\nu(M) = \{\bar{C}^q_\nu(M), \delta\}$, $\bar{C}^q_\nu(M) = C^q_\nu(M)/C^q_\nu(M) \cap C^q_0(M)$,

where $C^q_\nu(M) = \{\varphi \in C^q(M);\ \varphi(x^0, \ldots, x^q) = 0$ *if* $x^i = x^{i+1}$ *for some* $0 \leq i \leq q-1\}$;

(iii) $\bar{C}^*_\lambda(M) = \{\bar{C}^q_\lambda(M), \delta\}$, $\bar{C}^q_\lambda(M) = C^q_\lambda(M)/C^q_\lambda(M) \cap C^q_0(M)$,

where $C^q_\lambda(M) = \{\varphi \in C^q_\nu(M);\ \varphi(x^1, \ldots, x^q, x^0) = (-1)^q \varphi(x^0, x^1, \ldots, x^q)\}$.

Proof. They are obviously subcomplexes of $\bar{C}^*(M)$ and the cohomology of the associated presheaves is locally zero for $q > 0$. □

In order to find an explicit formula for a left inverse to $\bar{\rho}^*$, it will be helpful to bring into the discussion the universal complex of the Fréchet algebra $\mathscr{A} = C^\infty(M)$. We recall its definition: $\Omega^0(\mathscr{A}) = \mathscr{A}$, $\Omega^1(\mathscr{A}) = Ker(\mathscr{A} \hat{\otimes} \mathscr{A} \xrightarrow{\text{multipilication}} \mathscr{A})$, which is, in an obvious way, a bimodule over $\mathscr{A}$, and $\Omega^q(\mathscr{A}) = \Omega^1(\mathscr{A}) \hat{\otimes}_{\mathscr{A}} \dots \hat{\otimes}_{\mathscr{A}} \Omega^1(\mathscr{A})$ (q times) for $q \geq 1$; it is equipped with a continuous coboundary homomorphism $\partial : \Omega^q(\mathscr{A}) \to \Omega^{q+1}(\mathscr{A})$, uniquely characterized by the equations

$$\partial f = 1 \otimes f - f \otimes 1, \quad \forall f \in \mathscr{A},$$
$$\partial(f^0 \partial f^1 \otimes \dots \otimes \partial f^q) = \partial f^0 \otimes \partial f^1 \otimes \dots \otimes \partial f^q, \ \forall f^0, f^1, \dots, f^q \in \mathscr{A}.$$

There is a natural surjection ν of $C^q(M) \cong \mathscr{A} \hat{\otimes} \dots \hat{\otimes} \mathscr{A}$ ($q+1$ times) onto $\Omega^q(M)$, which sends an elementary tensor $f^0 \otimes f^1 \otimes \dots \otimes f^q \in C^q(M)$ to $f^0 \partial f^1 \otimes \dots \otimes \partial f^q \in \Omega^q(M)$. In particular, it sends

$$\delta(f^0 \otimes f^1 \otimes \dots \otimes f^q) = \sum_{i=0}^{q+1} (-1)^i f^0 \otimes \dots \otimes f^{i-1} \otimes 1 \otimes f^i \otimes \dots \otimes \dots \otimes f^q$$

to

$$\partial(f^0 \partial f^1 \otimes \dots \otimes \partial f^q) = \partial f^0 \otimes \partial f^1 \otimes \dots \otimes \partial f^q.$$

On the other hand, by the universality of $\{\Omega^*(\mathscr{A}), \partial\}$, there exists a canonical morphism of complexes $\mu : \Omega^*(\mathscr{A}) \to \Lambda^*(M)$ such that

$$\mu(f^0 \partial f^1 \otimes \dots \otimes \partial f^q) = f^0 \partial f^1 \wedge \dots \wedge \partial f^q.$$

Composing it with the morphism $\nu : C^*(M) \to \Omega^*(\mathscr{A})$, we obtain a morphism of complexes $\lambda = \mu \circ \nu : C^*(M) \to \Lambda^*(M)$, which evidently vanishes on $C_0^*(M)$. Thus, it induces a morphism $\bar{\lambda} : \bar{C}^*(M) \to \Lambda^*(M)$, characterized by:

$$\bar{\lambda}(f^0 \otimes f^1 \otimes \dots \otimes f^q)^- = f^0 \partial f^1 \wedge \dots \wedge \partial f^q.$$

For an arbitrary cochain $\bar{\varphi} \in \bar{C}^q(M)$ one has therefore:

$$\bar{\lambda}(\bar{\varphi})_x(v^1, \dots, v^q)$$
$$= \frac{1}{q!} \sum_{\tau \in \mathscr{S}_q} \operatorname{sgn}(\tau) \frac{\partial}{\partial \varepsilon_1} \dots \frac{\partial}{\partial \varepsilon_q} \varphi(x, \exp_x \varepsilon_1 v^{\tau(1)}, \dots, \exp_x \varepsilon_q v^{\tau(q)})|_{\varepsilon_i = 0}.$$

(1.5) LEMMA. *The morphism of complexes $\bar{\lambda} : \bar{C}^*(M) \to \Lambda^*(M)$ is a left inverse of $\bar{\rho}$: $\Lambda^*(M) \to \bar{C}^*(M)$ and it induces a two sided inverse $\bar{\lambda}^*$ of $\bar{\rho}^*$ in cohomology.*

Proof. According to the definitions of ρ and λ, if $\omega \in \Lambda^q(M)$

$$\lambda(\rho(\omega)) = \frac{1}{q!} \sum_{\tau \in \mathscr{S}_q} \operatorname{sgn}(\tau) \omega^\tau$$

where

$$\omega_x^\tau(v^1, \dots, v^q) = \frac{\partial}{\partial \varepsilon_1} \dots \frac{\partial}{\partial \varepsilon_q} \int_{s_q[x, \exp_x \varepsilon_1 v^{\tau(1)}, \dots, \exp_x \varepsilon_q v^{\tau(q)}]} \omega|_{\varepsilon_i = 0}.$$

To compute the right hand side, we can work in a neighborhood $B \in \mathscr{B}$ of x and assume, without loss of generality, that

$$\omega | B = f(u) du_{i_1} \wedge \dots \wedge du_{i_q},$$

in normal coordinates centered at x. Thus, if $(u_1(\mathbf{t}, \varepsilon), \ldots, u_m(\mathbf{t}, \varepsilon))$ are the coordinates of

$$u(\mathbf{t}, \varepsilon) = s_q[x, \exp_x \varepsilon v^{\tau(1)}, \ldots, \exp_x \varepsilon_q v^{\tau(q)}](\mathbf{t}) = t_0 x + \sum_{a=1}^{q} t_a \exp_x \varepsilon_a v^{\tau(a)},$$

one has:

$$\omega_x^{\tau}(v^1, \ldots, v^q) = \int_{\substack{t_1 + \ldots + t_q \leq 1 \\ 0 \leq t_1, \ldots, t_q \leq 1}} \frac{\partial}{\partial \varepsilon_1} \cdots \frac{\partial}{\partial \varepsilon_q}\left[f(u(\mathbf{t}, \varepsilon)) \det\left(\frac{\partial u_{i_a}}{\partial t_b}(\mathbf{t}, \varepsilon)\right)\right]\Big|_{\varepsilon_j = 0} dt_1 \wedge \ldots \wedge dt_q.$$

Recall now that $u(\mathbf{t}, \varepsilon)$ is the unique solution of the system:

$$\frac{\partial}{\partial u_i}\left[t_0 d^2(x, \exp_x u) + \sum_{a=1}^{q} t_a d^2(\exp_x \varepsilon_a v^{\tau(a)}, \exp_x u) \right] = 0, \; i = 1, \ldots, m.$$

Using the same expansion for d^2 as in (d'), but in normal coordinates centered at x, the system becomes:

$$t_0 u_i + \sum_{a=1}^{q} t_a \left[(u_i - \varepsilon_a v_i^{\tau(a)}) - \frac{1}{3}\varepsilon_a^2 \sum_{j,k,l} R_{ijkl}(x) v_j^{\tau(a)} v_l^{\tau(a)} (u_k - \varepsilon_a v_k^{\tau(a)}) \right]$$
$$+ \text{higher order terms} = 0, \quad i = 1, \ldots, m.$$

Differentiation with respect to ε_b at 0, gives

$$\frac{\partial u_i}{\partial \varepsilon_b}\Big|_{\varepsilon_b = 0} = t_b v_i^{\tau(b)},$$

therefore

$$u_i(\mathbf{t}, \varepsilon) = \sum_{a=1}^{m} t_a(\varepsilon_a v_i^{\tau(b)} + O(\varepsilon_a^2)).$$

Introducing this expression into the above formula of $\omega_x^{\tau}(v^1, \ldots, v^q)$ one obtains:

$$\omega_x^{\tau}(v^1, \ldots, v^q) = \int_{\substack{t_1 + \ldots + t_q \leq 1 \\ 0 \leq t_1, \ldots, t_q \leq 1}} f(x) \det(v_{i_a}^{\tau(b)}) dt_1 \wedge \ldots \wedge dt_q$$
$$= \frac{1}{q!} f(x) \det(du_{i_a}(v^{\tau(b)})) = \omega_x(v^{\tau(1)}, \ldots, v^{\tau(q)}).$$

Thus

$$\lambda(\rho(\omega))_x(v^1, \ldots, v^q) = \frac{1}{q!} \sum_{\tau \in \mathscr{S}_q} \operatorname{sgn}(\tau) \omega_x^{\tau}(v^1, \ldots, v^q)$$
$$= \frac{1}{q!} \sum_{\tau \in \mathscr{S}_q} \operatorname{sgn}(\tau) \omega_x(v^{\tau(1)}, \ldots, v^{\tau(q)}) = \omega_x(v^1, \ldots, v^q).$$

This proves the first part of the statement. The second part follows from Lemma (1.3). □

We conclude this section with a brief discussion of the Alexander–Spanier cohomology with compact supports. It can be defined as follows: $C_c^q(M)$ is, by definition, the subspace of $C_\infty^q(M)$ consisting of those $\varphi \in C_\infty^q(M)$ whose restriction to the complement $M - A$ of some relatively compact set A belongs to $C_0^q(M - A)$. Evidently, $C_{c,0}^q(M) \stackrel{def}{=} C_{\infty,0}^q(M)$ is contained in $C_c^q(M)$ and one considers the quotient $\bar{C}_c^q(M) = C_c^q(M)/C_{c,0}^q(M)$. The coboundary homomorphism δ is well-defined on $\bar{C}_c^q(M)$, giving a complex $\bar{C}_c^*(M)$. Its cohomology $\bar{H}_c^*(M)$ is the *Alexander–Spanier cohomology with compact supports of M*. This is of course, the smooth version, but again one can easily adapt the arguments in [32; Sec. 6.9] to show that $\bar{H}_c^*(M)$ is naturally isomorphic to the singular cohomology with compact supports $H_c^*(M)$.

It is obvious that $\bar{\lambda}$ maps $\bar{C}_c^*(M)$ to $\Lambda_c^*(M) =$ the de Rham complex with compact supports, and it is also clear that $\bar{\rho}$ maps $\Lambda_c^*(M)$ to $\bar{C}_c^*(M)$. Since $\bar{\lambda} \circ \bar{\rho} = Id$, it follows that $\bar{\lambda}$ is surjective and $\bar{\rho}$ is injective. The same is true about the induced homomorphisms

$$\bar{\lambda}^* : \bar{H}_c^*(M) \to H_{dR,c}^*(M), \text{ resp. } \bar{\rho}^* : H_{dR,c}^*(M) \to \bar{H}_c^*(M).$$

In particular, if the cohomology with compact supports of M is finite dimensional, $\bar{\lambda}^*$ and $\bar{\rho}^*$ are seen to be isomorphisms. We leave to the diligent reader the task of proving this fact for an arbitrary C^∞ manifold. We mention though the following consequence, which will be helpful in the next section.

(1.6) REMARK. *The cohomology $\bar{H}_c^*(M)$ can be computed by means of the complex $\{\bar{C}_{\lambda,cc}^*(M), \delta\}$, where*

$$C_{\lambda,cc}^q(M) = C_{\lambda,c}^q(M) \cap C_c^\infty(M^{q+1})$$

and

$$\bar{C}_{\lambda,cc}^q(M) = C_{\lambda,cc}^q(M)/C_{\lambda,cc}^q(M) \cap C_0^q(M).$$

§2. LOCALIZED ANALYTIC INDICES

As a refinement of the analytic index of an elliptic symbol, we shall construct for each even-dimensional Alexander–Spanier cohomology class (with compact support) on a C^∞ manifold M a *localized index* map from the K-group $K^0(T^*M, T^*M - M)$ to $\mathbb{C}$. When M is compact, $K^0(T^*M, T^*M - M)$ is the K-theory with compact support $K_c^0(T^*M)$ and the ordinary index map corresponds to the unit class $[1] \in \bar{H}^0(M)$.

We shall need, and freely use, several basic facts of pseudo-differential calculus on manifolds, strictly included among those employed in [3, I]. Here is some related notation which will be used in what follows. Given two C^∞ complex vector bundles E, F over M we denote by $\Psi^r(M, E, F)$ the set of order r classical pseudo-differential operators $A: C_c^\infty(M, E \otimes |\Lambda|^{1/2}(M)) \to C^\infty(M, F \otimes |\Lambda|^{1/2}(M))$, where $|\Lambda|^\alpha(M)$ stands for the line bundle of α-densities on M. Its inclusion in the definition has the effect that the distributional kernel $A(x, y)$ of $A \in \Psi^r(M; E, F)$ is a section of the bundle $Hom(E, F) \otimes |\Lambda|^{1/2}(M \times M)$, i.e.

$$A(x, y) \in Hom(E_y, F_x) \otimes |\Lambda|^{1/2} T_x M \otimes |\Lambda|^{1/2} T_y M, \ \forall (x, y) \in M \times M.$$

The subset of differential operators is denoted $DO^r(M; E, F)$ and the subset of elliptic operators is denoted $\Psi^r(M; E, F)^{-1}$. By $Psy^r(M; E, F)$ we denote the set of "principal symbols" of order r, i.e. the subspace of $C^\infty(T^*M - M, Hom(\pi^*E, \pi^*F))$, where π is the projection onto M, formed of sections which are homogeneous of degree r. The subset of elliptic symbols is denoted $Psy^r(M; E, F)^{-1}$. When $E = F$ we shall suppress the second bundle from the notation and when $E = F =$ the trivial bundle we shall omit the bundle altogether.

Let $A^0, \ldots, A^q \in \Psi^\infty(M; E)$ with at least one of them in $\Psi^{-\infty}(M; E)$; we define the distribution $tr(A^0, \ldots, A^q)$ on M^{q+1} by the formula

$$tr(A^0, \ldots, A^q)(\varphi) = (-1)^q \int tr(A^0(x^0, x^1) \ldots A^q(x^q, x^0))\varphi(x^0, \ldots, x^q), \ \forall \varphi \in C_c^\infty(M^{q+1}).$$

For a fixed $\varphi \in C_c^\infty(M^{q+1})$, we also set

$$\tau(\varphi)(A^0, \ldots, A^q) = tr(A^0, \ldots, A^q)(\varphi), \ \forall A^j \in \Psi^\infty(M; E).$$

(2.1) LEMMA. (i) *Let $\varphi \in C^q_{\lambda, cc}(M)$. Then $\tau(\varphi) \in C^q_\lambda(\Psi^{-\infty}(M; E))$ = the space of q-dimensional cyclic cochains of the algebra $\Psi^{-\infty}(M; E)$, i.e.*

$$\tau(\varphi)(A^1, \ldots, A^q, A^0) = (-1)^q \tau(\varphi)(A^0, \ldots, A^{q-1}, A^q) \quad \forall A^j \in \Psi^{-\infty}(M; E).$$

(ii) *$\tau: C^*_{\lambda, c}(M) \to C^*_\lambda(\Psi^{-\infty}(M; E))$ is a homomorphism of complexes, i.e.*

$$\tau(\delta\varphi) = b\tau(\varphi),$$

where b is the coboundary of the cyclic cohomology complex [8].

(iii) *If $q > 0$, $A^j \in \Psi^{-\infty}(M; E)$ and $f^j \in DO^0(M; E)$, $j = 0, \ldots, q$, one has*

$$\tau(\varphi)(A^0 + f^0, \ldots, A^q + f^q) = \tau(\varphi)(A^0, \ldots, A^q).$$

Proof. The first property follows from the cyclicity of the trace and the skew cyclicity of φ. The second can be checked by a straightforward calculation. Finally, (iii) is a consequence of the fact that φ vanishes along diagonals $x^j = x^{j+1}$. □

To motivate our definition of the localized indices, let us recall, in an operator algebraic language, the definition of the ordinary analytic index map for M compact. One starts with the exact sequence of C^*-algebras

$$0 \to \mathscr{K}_M \xrightarrow{\iota} \mathscr{L}_M \xrightarrow{\sigma} C(S^*M) \to 0,$$

where $\mathscr{K}_M$ = the algebra of compact operators on $L^2(M; |\Lambda|^{1/2}(M))$, $\mathscr{L}_M$ = the norm closure of $\Psi^0(M)$ in the algebra of all bounded operators on $L^2(M; |\Lambda|^{1/2}(M))$, S^*M is the unit cosphere bundle corresponding to a Riemannian metric, ι is the inclusion and σ is obtained from the principal symbol map $\sigma_{pr}: \Psi^0(M) \to C^\infty(T^*M - M)$ by restricting the symbols to S^*M. This sequence gives rise to a six term exact sequence of K-groups:

$$\begin{array}{ccccc} K_1(\mathscr{K}_M) & \xrightarrow{\iota_1} & K_1(\mathscr{L}_M) & \xrightarrow{\sigma_1} & K_1(C(S^*M)) \\ 0\uparrow & & & & \downarrow\partial \\ K_0(C(S^*M)) & \xleftarrow{\sigma_0} & K_0(\mathscr{L}_M) & \xleftarrow{\iota_0} & K_0(\mathscr{K}_M). \end{array}$$

Since $K_1(\mathscr{K}_M) = K_1(\mathbb{C}) = 0$, one of the two connecting maps is trivial. The other one, denoted ∂, gives the analytic index map *Index*: $K_c^0(T^*M) \to \mathbb{Z}$ of [3; I, §6], after the canonical identifications $K_1(C(S^*M)) \cong K_c^0(T^*M)$ and $K_0(\mathscr{K}_M) \cong K_0(\mathbb{C}) \cong \mathbb{Z}$. More explicitly, the general algebraic definition of the connecting map in K-theory leads to the following recipe for constructing the analytic index of an elliptic symbol $a \in Psy^0(M; E, F)^{-1}$. The symbol $\tilde{a} = \begin{pmatrix} 0 & -a^{-1} \\ a & 0 \end{pmatrix}$ defines, after embedding E in a trivial bundle, an element of $GL_N^0(C^\infty(S^*M))$ (where the superscript 0 indicates the connected component of identity) for N sufficiently large. It can, therefore, be lifted to $GL_N^0(\Psi^0(M))$. For example, if one chooses $A \in \Psi^0(M; E, F)$ with $\sigma_{pr}(A) = a$ and $B \in \Psi^0(M; F, E)$ with $\sigma_{pr}(B) = a^{-1}$, then $S_0 = I - BA \in \Psi^{-1}(M; E)$ and $S_1 = I - AB \in \Psi^{-1}(M; F)$, hence

$$L = \begin{pmatrix} S_0 & -B - S_0 B \\ A & S_1 \end{pmatrix} \in \Psi^0(M; E \oplus F)$$

defines such a lift. By definition,

$$\partial([a]) = [P] - [e],$$

where P and e are idempotents defined as follows:

$$P = L\begin{pmatrix} I_E & 0 \\ 0 & 0 \end{pmatrix}L^{-1}, \qquad e = \begin{pmatrix} 0 & 0 \\ 0 & I_F \end{pmatrix}.$$

The definition of $\partial([a])$ can be shown to be independent of the lift L. In particular, one may improve, at no extra cost, the choice of the parametrix B such that S_0 *and* S_1 *are smoothing operators.* Then

$$R = P - e = \begin{pmatrix} S_0^2 & S_0(I + S_0)B \\ S_1 A & -S_1^2 \end{pmatrix} \in \Psi^{-\infty}(M; E \oplus F)$$

and one has

$$Tr R = Tr S_0^2 - Tr S_1^2 = \text{Index } A,$$

which explains why ∂ can be regarded as the analytic index map.

Evidently, the analytical index map does not capture fully the *local* information carried by the symbol. It disregards for instance the possibility of localizing at will, around the diagonal, the above construction. By taking advantage of this important feature, we shall construct a pairing of the above projections with arbitrary Alexander–Spanier cocycles on M, which will recapture the stable information carried by the symbols.

We now proceed to describe this pairing. Since the compactness of the manifold is not really relevant to this point, we shall drop this assumption and replace it with the use of cohomology with compact supports. Consider a cocycle $\varphi \in Z^q_{\lambda, cc}(M)$, that is $\varphi \in C^q_{\lambda, cc}(M)$ and $\delta\varphi \in C_0^{q+1}(M)$. Let L be an invertible lift of the symbol $\tilde{a}$ such that $L(x, y) = 0$ outside a "small" neighborhood of the diagonal, the "size" of which depends on where $\delta\varphi$ vanishes, in a way which will be obvious from the context. Such a lift can be manufactured, for example, by localizing the support of A and B in the above construction. Denoting as before,

$$P_L = L\begin{pmatrix} I_E & 0 \\ 0 & 0 \end{pmatrix}L^{-1}, \qquad R_L = P_L - \begin{pmatrix} 0 & 0 \\ 0 & I_F \end{pmatrix} \in \Psi^{-\infty}$$

we define

$$Ind_\varphi(a) = \tau(\varphi)(R_L, \ldots, R_L) = (-1)^q \int_{M^{q+1}} tr(R_L(x^0, x^1) \ldots R_L(x^q, x^0))\varphi(x^0, \ldots, x^q)$$

One has to check that the definition makes sense, i.e. that the right hand side is independent of the lift L. If q is odd, in view of Lemma (2.1) (i), this is obvious; it is also uninteresting, since it gives $Ind_\varphi(a) = 0$. So, we shall assume from now on that q is even.

(2.2) Lemma. *Let* $\{P_s; s \in [0, 1]\} \subset DO^0 + \Psi^{-\infty}$ *be a one-parameter family of idempotents piecewise* C^1 *(as a map from* $[0, 1]$ *to* $Op^0(M; E, F)$*, topologized as in* [3, I, §5]).

Then

$$\tau(\varphi)(P_1, \ldots, P_1) - \tau(\varphi)(P_0, \ldots, P_0) = (q+1)\int_0^1 \tau(\delta\varphi)(T_s, P_s, \ldots, P_s)\,ds,$$

where $T_s = (1 - 2P_s)\dfrac{d}{ds}P_s$.

Proof. We notice that $\dfrac{d}{ds}P_s = [T_s, P_s]$ and therefore,

$$\begin{aligned}\frac{d}{ds}\tau(\varphi)(P_s, \ldots P_s) &= \sum_0^q \tau(\varphi)(P_s, \ldots, [T_s, P_s], \ldots, P_s)\\ &= (q+1)\tau(\varphi)([T_s, P_s], P_s, \ldots, P_s).\end{aligned}$$

This last expression is easily recognized to coincide with $(q+1)b\tau(\varphi)(T_s, P_s, \ldots, P_s)$, which by Lemma (2.1) (ii) is in turn equal to $(q+1)\tau(\delta\varphi)(T_s, P_s, \ldots, P_s)$. ☐

We can now show that $Ind_\varphi(a)$ is well-defined. Note that if $q = 0$ and $\varphi \not\equiv 0$ then M must be compact, in which case this follows from the fact that $\partial([a])$ is well-defined (see, e.g. [6, Sec. 8.3]). Thanks to Lemma (2.2), essentially the same proof will work for $q > 0$. Indeed, let L_0, L_1 be two invertible lifts of $\tilde{a} = \begin{pmatrix} 0 & -a^{-1} \\ a & 0 \end{pmatrix}$ as above, with support sufficiently localized around the diagonal. Let $P_i = L_i\begin{pmatrix} I_E & 0 \\ 0 & 0 \end{pmatrix}L_i^{-1}$ where $i = 0, 1$. By Lemma (2.1) (iii), it suffices to check that

$$\tau(\varphi)(P_0, \ldots, P_0) = \tau(\varphi)(P_1, \ldots, P_1).$$

Each member of this identity remains unchanged if one replaces P_i by $\tilde{P}_i = \begin{pmatrix} P_i & 0 \\ 0 & 0_{E \oplus F} \end{pmatrix}$, the advantage being that one can now employ a well-known recipe to construct a path of idempotents joining $\tilde{P}_0$ and $\tilde{P}_1$. Namely, one first defines a path of invertibles by setting

$$J_s = \begin{pmatrix} L_1 L_0^{-1} & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} \cos\frac{\pi s}{2} & -\sin\frac{\pi s}{2} \\ \sin\frac{\pi s}{2} & \cos\frac{\pi s}{2} \end{pmatrix}\begin{pmatrix} L_0 L_1^{-1} & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} \cos\frac{\pi s}{2} & \sin\frac{\pi s}{2} \\ -\sin\frac{\pi s}{2} & \cos\frac{\pi s}{2} \end{pmatrix}$$

and then one defines $\tilde{P}_s = J_s \tilde{P}_0 J_s^{-1}$, $\forall s \in [0, 1]$. Since in the process the support has been maintained localized, Lemma (2.2) gives the desired equality.

Evidently, if φ is altered by adding a locally zero cochain $Ind_\varphi(a)$ remains unchanged. Thus, we can in fact set, for $a \in Psy^0(M; E, F)^{-1}$,

$$Ind_{\bar{\varphi}}(a) = Ind_\varphi(a)$$

and the definition is unambiguous. The next lemma shows that the *localized* index map thus defined actually depends only on the cohomology class $[\bar{\varphi}]$.

(2.3) LEMMA. *If* $\psi \in C^{q-1}_{\lambda, cc}(M)$, *then* $Ind_{\delta\psi}(a) = 0$.

Proof. With $P = P_L$ and L as above, one has:

$$\begin{aligned}Ind_{\delta\psi}(a) &= \tau(\delta\psi)(P, \ldots, P) = b\tau(\psi)(P, \ldots, P)\\ &= \tau(\psi)(P, \ldots, P) = -\tau(\psi)(P, \ldots, P);\end{aligned}$$

here we made use of (2.1) (i)–(ii) and of the fact that q is even. □

Recalling Remark (1.6), we conclude that we have defined, for each class $[\bar{\varphi}] \in \bar{H}_c^{ev}(M)$, an index map $Ind_{[\bar{\varphi}]}$ on the set of all elliptic principal symbols (of order 0). It remains to show that this map descends to K-theory.

(2.4) THEOREM. *For any $[\bar{\varphi}] \in \bar{H}_c^{ev}(M)$ the map $Ind_{[\bar{\varphi}]}$ from elliptic symbols to $\mathbb{C}$ induces a homomorphism $Ind_{[\bar{\varphi}]}: K^0(T^*M, T^*M - M) \to \mathbb{C}$.*

Proof. For the convenience of the reader, we briefly review the definition of the relative K-group $K^0(T^*M, T^*M - M)$ in the spirit of [3, I]. Consider the set $\mathscr{C}(T^*M, T^*M - M)$ of triples (σ, E, F) where E, F are C^∞ complex vector bundles over T^*M and $\sigma: E|_{T^*M - M} \to F|_{T^*M - M}$ is an isomorphism. Two such triples (σ, E, F) and (σ', E', F') are *isomorphic* if there are isomorphisms $\alpha: E \to E'$, $\beta: F \to F'$ such that $\sigma' \circ \alpha = \beta \circ \sigma$ over $T^*M - M$. Two triples (σ^i, E^i, F^i), $i = 0, 1$, are called *homotopic* if there exists an element $(\sigma, E, F) \in \mathscr{C}(T^*M \times [0,1], (T^*M - M) \times [0,1])$ such that (σ^i, E^i, F^i) is isomorphic to $(\sigma, E, F)|T^*M \times \{i\}$, $i = 0, 1$. The set of homotopy classes, $C = C(T^*M, T^*M - M)$, equipped with the direct sum operation, forms an abelian semigroup. It contains a sub-semigroup $C_\phi = C_\phi(T^*M, T^*M - M)$ represented by triples (σ, E, F) with $\sigma: E \to F$ an isomorphism defined everywhere. The quotient semigroup C/C_ϕ is actually a group, which can be taken as the definition of $K^0(T^*M, T^*M - M)$.

In particular each elliptic principal symbol $a \in Psy^0(M; E, F)^{-1}$ defines an element (a, π^*E, π^*F) in $\mathscr{C} = \mathscr{C}(T^*M, T^*M - M)$, homogeneous of degree 0. Let $\mathscr{C}^0$ be the subset of all such elements in $\mathscr{C}$. It is well-known and easy to see that the set of homotopy classes in $\mathscr{C}^0$ exhausts the whole group $K^0(T^*M, T^*M - M)$. Indeed, since M is a deformation retract of T^*M (which is paracompact), any $(\sigma, E, F) \in \mathscr{C}$ is isomorphic to a triple $(a_\sigma, \pi^*E_0, \pi^*F_0) \in \mathscr{C}^0$, where E_0, F_0 are the restrictions of E, F to the zero-section and a_σ is extended by homogeneity of degree 0 from the restriction of σ to the co-sphere bundle $S^*M = \{\xi \in T^*M, \|\xi\| = 1\}$ (corresponding to a Riemannian metric on M). Moreover, this isomorphism is unique up to homotopy if the isomorphisms $E \cong \pi^*E_0$, $F \cong \pi^*F_0$ are chosen to be the identity on the zero-section.

The map $Ind_{[\bar{\varphi}]}$ being already defined on $\mathscr{C}^0$, all we need to check is that if $(a, \pi^*E, \pi^*F) \in \mathscr{C}^0(T^*M \times [0,1], (T^*M - M) \times [0,1])$ then $Ind_{[\bar{\varphi}]}$ is constant along the path $\{a_s\}$ where a_s corresponds to $(a, \pi^*E, \pi^*F)|(T^*M - M) \times \{s\}$, $s \in [0,1]$. Since M is paracompact, one has $E \cong E_0 \times [0,1]$, where $E_0 = E|M \times \{0\}$ and similarly $F \cong F_0 \times [0,1]$ (cf. [22; Cor. 4.5]), so that $a_s \in Psy^0(M; E_0, F_0)^{-1}$, $\forall s \in [0,1]$. Let $\{A_s\}$ (resp. $\{B_s\}$) be a C^1 path in $\Psi^0(M; E_0, F_0)$ (resp. $\Psi^0(M; F_0, E_0)$) such that $\sigma_{pr}(A_s) = a_s$, $A_sB_s - I \in \Psi^{-\infty}$, $B_sA_s - I \in \Psi^{-\infty}$ and each A_s (resp. B_s) is supported in a sufficiently small neighborhood of the diagonal. Denote by L_s the lifting of $\tilde{a}_s$ manufactured from A_s and B_s, and by P_s the corresponding idempotent. Again, we may assume $q > 0$, and then the claim follows by applying Lemma (2.2) to the path $\{P_s\}$. □

The localized index maps thus defined can be easily transferred to elliptic operators. Namely, if $[\bar{\varphi}] \in \bar{H}_c^{ev}(M)$ and $A \in \Psi^r(M; E, F)^{-1}$, we define

$$Ind_{[\bar{\varphi}]}A = Ind_{[\bar{\varphi}]}(a),$$

where $a \in Psy^0(M; E, F)$ is uniquely determined by the condition $a|S^*M = \sigma_{pr}(A)|S^*M$. As in the case of the ordinary index, it will be useful to relate these indices to heat operators. Since for the computation of a $[\bar{\varphi}]$-index we can always pick a representative φ with compact support, there will be no loss of generality in restricting our attention to compact (oriented, even-dimensional) manifolds M. (We shall elaborate a bit on this point later). Furthermore, since for M compact, $K_c^0(T^*M)$ is generated modulo 2-torsion by signature-type symbols (see [2, §7]), it will suffice to consider elliptic differential operators (actually generalized signature operators would already be enough).

So let us assume that M is compact and that $D \in DO^r(M; E, F)^{-1}$. Then

$$Q(D) = \frac{I - e^{-\frac{1}{2}D^*D}}{D^*D} D^* \in \Psi^{-r}(M; F, E)$$

is a parametrix for D. Indeed, one has

$$S_0 = I - QD = e^{-\frac{1}{2}D^*D}, \qquad S_1 = I - DQ = e^{-\frac{1}{2}DD^*}.$$

The corresponding idempotent has the expression:

$$P(D) = \begin{pmatrix} e^{-D^*D} & e^{-\frac{1}{2}D^*D} \dfrac{I - e^{-D^*D}}{D^*D} D \\ e^{-\frac{1}{2}DD^*} D & I - e^{-DD^*} \end{pmatrix}.$$

Via the one-parameter family of idempotents

$$P_s(D) = \begin{pmatrix} e^{-D^*D} & e^{-\frac{1}{2}D^*D} \left(\dfrac{I - e^{-D^*D}}{D^*D} \right)^{\frac{1}{2}+s} D^* \\ e^{-\frac{1}{2}DD^*} \left(\dfrac{I - e^{-DD^*}}{D^*D} \right)^{\frac{1}{2}-s} D & I - e^{-DD^*} \end{pmatrix}$$

where $s \in [0, \frac{1}{2}]$, $P(D)$ is seen to be homotopic to the self-adjoint idempotent (found by A. Wasserman [35]):

$$W(D) = \begin{pmatrix} e^{-D^*D} & e^{-\frac{1}{2}D^*D} \left(\dfrac{I - e^{-D^*D}}{D^*D} \right)^{\frac{1}{2}} D^* \\ e^{-\frac{1}{2}DD^*} \left(\dfrac{I - e^{-DD^*}}{D^*D} \right)^{\frac{1}{2}} D & I - e^{-DD^*} \end{pmatrix}.$$

Passing to heat operators we have lost, of course, the localization property. Fortunately, this can be remedied by replacing D with the family tD, $t > 0$, and letting $t \to 0$. More exactly, one has the following "localized" version of the McKean–Singer formula.

(2.5) LEMMA. *Let* $W(t) = W(tD)$, $W'(t) = W(t) - \begin{pmatrix} 0 & 0 \\ 0 & I \end{pmatrix}$, $t > 0$. *Then, for any* $\tilde{\varphi} \in \tilde{Z}_\lambda^q(M)$ *one has*

$$\tau(\varphi)(W'(t), \ldots, W'(t)) = Ind_\varphi D + O(t^\infty).$$

Proof. When $q = 0$ the statement follows from the classical McKean–Singer formula. We may therefore assume that $q > 0$.

Choose $\alpha \in C^\infty(M \times M)$ with support contained in a sufficiently small neighborhood of the diagonal (depending on $\delta\varphi$), such that $\alpha \equiv 1$ on some neighborhood of the diagonal and $\alpha \geq 0$ everywhere. Define, for $t > 0$, the operator $\tilde{Q}(t) \in \Psi^{-r}(M; F, E)$ by

$$\tilde{Q}(t)(x, y) = \alpha(x, y) Q(tD)(x, y).$$

Since $\tilde{Q}(t) - Q(tD) \in \Psi^{-\infty}(M; F, E)$, each $\tilde{Q}(t)$ is a parametrix for tD with support contained in the support of α. Thus, if we denote by $\tilde{P}(t)$ the corresponding idempotent, one has

$$\tau(\varphi)(\tilde{P}(t), \ldots, \tilde{P}(t)) = Ind_\varphi tD = Ind_\varphi D.$$

Denote $S_0(t) = I - Q(tD)tD$, $S_1(t) = I - tDQ(tD)$ and, correspondingly,

$$\tilde{S}_0(t) = I - \tilde{Q}(t)tD, \; \tilde{S}_1(t) = I - tD\tilde{Q}(t).$$

One easily checks that

$$S_i(t)(x, y) - \tilde{S}_i(t)(x, y) = O(t^\infty), \qquad i = 0, 1$$

in the C^∞ topology. Thus, with $P(t) = P(tD)$, one has

$$P(t)(x, y) - \tilde{P}(t)(x, y) = O(t^\infty).$$

On the other hand, $P(t)(x, y) = O(t^{-\nu})$, for some fixed $\nu > 0$, as can be seen, for instance, by regarding tD as a pseudo-differential family in the sense of Widom (cf. [33], [34]). Therefore,

$$\tau(\varphi)(P(t), \ldots, P(t)) = \tau(\varphi)(\tilde{P}(t), \ldots, \tilde{P}(t)) + O(t^\infty) = Ind_\varphi D + O(t^\infty).$$

To pass from $P(t)$ to $W(t)$, we apply Lemma (2.2) to the path $s \to P_s(t) = P_s(tD)$, $s \in [0, 1/2]$. One obtains

$$\tau(\varphi)(P(t), \ldots, P(t)) - \tau(\varphi)(W(t), \ldots, W(t)) =$$
$$(q + 1) \int_0^{1/2} \tau(\delta\varphi)(T_s(t), P_s(t), \ldots, P_s(t))\, ds$$

and thus it remains to show that the larger integral is an $O(t^\infty)$.

Since $\delta\varphi$ is cyclic we can replace each $P_s(t)$ by $R_s(t) = P_s(t) - \begin{pmatrix} 0 & 0 \\ 0 & I \end{pmatrix}$. Furthermore, since $\delta\varphi$ is locally zero we only need to estimate

$$\int_{C(\rho)} tr(T_s(t)(x^0, x^1) R_s(t)(x^1, x^2) \ldots R_s(t)(x^{q+1}, x^0))$$

where $C(\rho) = \{(x^0, \ldots, x^{q+1}); d^2(x^0, x^1) + \ldots + d^2(x^{q+1}, x^0) \geq \rho\}$ with $\rho > 0$ fixed. In turn, the above integral is a sum of 2^{q+1} integrals of the form

$$\int_{C(\rho)} tr(A_s^0(t)(x^0, x^1) \ldots A_s^{q+1}(t)(x^{q+1}, x^0)),$$

where for each $0 \leq i \leq q + 1$, $A_s^i = f_s^i(t^2\Delta)$ with Δ either D^*D or DD^* and f_s^i a Schwartz function on $\mathbb{R}^+$ which has a holomorphic extension to a domain containing $\mathbb{R}^+$. On the other hand, since $C(\rho) \subset \bigcup_{i=0}^{q+1} C^i(\rho)$, where

$$C^i(\rho) = \left\{(x^0, \ldots, x^{q+1}); d^2(x^i, x^{i+1}) \geq \frac{\rho}{q+2}\right\}, \qquad i = 0, \ldots, q,$$

$$C^{q+1}(\rho) = \left\{(x^0, \ldots, x^{q+1}); d^2(x^{q+1}, x^0) \geq \frac{\rho}{q+2}\right\},$$

it is enough to estimate the integrals

$$\int_{C^i(\rho)} tr(f_s^0(t^2\Delta)(x^0, x^1) \ldots f_s^{q+1}(t^2\Delta)(x^{q+1}, x^0)).$$

By Cauchy–Schwartz, the square of the absolute value of such an expression is majorized by

$$\int_{d^2(x^i, x^{i+1}) \geq \frac{\varepsilon}{q+2}} |f_s^i(t^2\Delta)(x^i, x^{i+1})|^2 \prod_{j \neq i} \|f_s^j(t^2\Delta)\|_{HS}^2.$$

Now tD is a (pseudo) differential family in the sense of Widom, so we can invoke [33, §§5–6] (see also [34, §5]) for the estimates:

$$\prod_{j \neq i} \|f_s^j(t^2\Delta)\|_{HS}^2 = O(t^{-\nu}), \qquad \text{for some fixed } \nu > 0$$

and

$$\int_{d^2(x,y) \geq \frac{\varepsilon}{q+2}} |f^i(t^2\Delta)(x,y)|^2 = O(t^\infty),$$

uniformly for $s \in [\frac{1}{2}, 1]$. □

§3. THE LOCALIZED INDEX FORMULA

We shall now use the heat equation approach, in Getzler's improved version [14], to find the topological expression of the localized analytic indices of elliptic operators. Actually, we shall prove a stronger, local formula which establishes the desired equality directly at the level of forms, belonging to $\Omega^* A$ on one side and to $\Lambda^*(M)$ on the other side.

The "localization" of the problem will also enable us to reduce it, as usual, to the case of twisted Dirac operators. Therefore, we shall temporarily assume that M is compact, even dimensional, oriented and equipped with a spin structure. We denote by $S = S^+ \oplus S^-$ the corresponding spin bundle, endowed with the standard metric and connection. Given any complex vector bundle E over M, with metric and connection ∇_E, we denote by $\not{D}_E$, or simply $\not{D}$, the corresponding Dirac operator, acting on the sections of $S \otimes E$. Since M has an underlying Riemannian structure, which in particular gives a volume element, we now stop using the bundle of half densities and revert to the more standard notation. We adopt, however, the Clifford algebra conventions in [14]; in particular $\not{D}_E^* = -\not{D}_E$.

Given $\varphi \in Z_\lambda^{2q}(M)$, we want to compute

$$Ind_\varphi \not{D}_E^+ = \lim_{t \to 0} \tau(\varphi)(W'(t\not{D}), \ldots, W'(t\not{D}))$$

(cf. Lemma (2.5)), where

$$W'(t\not{D}) = (e^{t^2\not{D}^2} + e^{\frac{1}{2}t^2\not{D}^2} w(-t^2\not{D}^2)t\not{D})\gamma, \ \gamma = \begin{pmatrix} I_{S^+ \oplus E} & 0 \\ 0 & -I_{S^- \oplus E} \end{pmatrix}$$

and

$$w(x) = \begin{cases} \left(\dfrac{1 - e^{-x}}{x}\right)^{1/2}, & x > 0 \\ 1, & x = 0 \end{cases}$$

Regarding now φ as an element of $C_\nu^{2q}(M) = \Omega^{2q}(\mathscr{A})$ (see §1), we can write it as a sum of a series

$$\varphi = \sum_{\mu=1}^{\infty} f_\mu^0 \partial f_\mu^1 \otimes \ldots \otimes \partial f_\mu^{2q}, \ f_\mu^j \in \mathscr{A} = C^\infty(M)$$

which converges absolutely in the Fréchet topology of $\Omega^{2q}(\mathscr{A})$. Thus, it will be sufficient to compute

$$I(\psi; D\!\!\!/) = \lim_{t \to 0} \tau(\psi)(W'(tD\!\!\!/), \dots, W'(tD\!\!\!/))$$

for an elementary tensor

$$\psi = f^0 \partial f^1 \otimes \dots \otimes \partial f^{2q}, \qquad f^j \in \mathscr{A}.$$

Note that there is no *a priori* reason for the above limit to exist and it is rather remarkable that it can be calculated, as we shall see below.

It is easy to check that if $A^0, \dots, A^{2q} \in \Psi^{\infty}(M)$ then

$$\tau(\psi)(A^0, \dots, A^{2q}) = Tr(A^{2q} f^0 [A^0, f^1] \dots [A^{2q-1}, f^{2q}]).$$

Therefore, the quantity to be computed is

$$I(\psi, D\!\!\!/) = \lim_{t \to 0} Tr_s((e^{t^2 D\!\!\!/^2} - e^{\frac{1}{2}t^2 D\!\!\!/^2} w(-t^2 D\!\!\!/^2) t D\!\!\!/) f^0 [W'(t), f^1] \dots [W'(t), f^{2q}]),$$

where $Tr_s = Tr \circ \gamma$ is the *supertrace*. We shall use Getzler's symbolic calculus [14] to find this limit.

Let $\Pi(t)$ denote the operator under Tr_s. In the notation of [14], $\Pi(t) \in Op\mathscr{S}^{-\infty}(E)$. In particular, Theorem 3.7 in [14] applies to give:

$$Tr_s \Pi(t) = (2\pi)^{-2n} \int_{T^*M} tr_s \sigma_{t^{-1}}(\Pi(t))(x, \xi)\, dx\, d\xi, \quad t > 0;$$

here $2n = \dim M$, tr_s is the pointwise supertrace, σ denotes the Getzler symbol map [14; p. 170] from $Op\mathscr{S}^{-\infty}(E)$ to $\mathscr{S}^{\infty}(E)$ $(= S^{\infty}(S \otimes E)$, equipped with the Getzler filtration [14; p. 169]), $\sigma_{t^{-1}}$ is the rescaled symbol [14; (3.1)] and $dx\,d\xi$ is the symplectic measure on T^*M.

In order to organize the computation, it will be convenient to introduce an auxiliary algebra $\mathscr{A}(D\!\!\!/)$. By definition, $\mathscr{A}(D\!\!\!/)$ is the algebra generated (in a strictly algebraic sense) by $\mathscr{A}, \gamma, D\!\!\!/, (\lambda + D\!\!\!/^2)^{-1}, \lambda \in \mathbb{C} - [0, \infty[$, and operators of the form $u(-D\!\!\!/^2)$ with u a Schwartz function on $[0, \infty[$ which admits a holomorphic extension to a complex neighborhood of $[0, \infty[$. The elements of $\mathscr{A}(D\!\!\!/)$ yield (possibly unbounded) operators on $L^2(M; S \otimes E)$, which also belong to $Op\mathscr{S}^{\infty}(E)$. If $A \in \mathscr{A}(D\!\!\!/)$, we let $A(t) = \rho_t(A)$ be the operator obtained by replacing $D\!\!\!/$ with $tD\!\!\!/$, $t > 0$, in the expression of A. For example, the operator $\Pi(t)$ introduced above corresponds to

$$\Pi = (e^{D\!\!\!/^2} - e^{\frac{1}{2}D\!\!\!/^2} w(-D\!\!\!/^2) D\!\!\!/) f^0 [W', f^1] \dots [W', f^{2q}]$$

where

$$W' = (e^{D\!\!\!/^2} - e^{\frac{1}{2}D\!\!\!/^2} w(-D\!\!\!/^2) D\!\!\!/)\gamma.$$

Since $\rho_t(\mathscr{A}(D\!\!\!/)) \subset Op\mathscr{S}^{\infty}(E)$, any operator $A \in \mathscr{A}(D\!\!\!/)$ has a well-defined *order* (with respect to the Getzler filtration on $Op\mathscr{S}^{\infty}(E)$). We introduce the notion of *asymptotic order* for elements of $\mathscr{A}(D\!\!\!/)$ as follows. If A belongs to the subalgebra $\mathscr{A}_{diff}(D\!\!\!/)$ generated by $\mathscr{A}, \gamma$ and $D\!\!\!/$, we form $A(t)$, assign to t the order -1, then define the asymptotic order of A as being the total order of $A(t)$. In particular $D\!\!\!/^2$ has asymptotic order 0. We give the same asymptotic order 0 to any function of $-D\!\!\!/^2$. Finally, we extend the definition to the whole $\mathscr{A}(D\!\!\!/)$ in the usual fashion. We shall use superscripts to indicate the Getzler order and subscripts to indicate the asymptotic order. Thus $\mathscr{A}^r_k(D\!\!\!/)$ means the subspace of $\mathscr{A}(D\!\!\!/)$ consisting of operators of Getzler order r and asymptotic order k.

The following result is the "Fundamental Lemma" of Getzler's symbolic calculus. Although not explicitly stated, it is practically proved and implicitly used in [14].

(3.1) LEMMA. (i) *Let* $A \in \mathscr{A}_0^r(\not{D})$. *Then, if* $A \in Op\mathscr{S}^r(E)$,

$$\sigma_{t^{-1}}(A(t)) = a + O(t), \qquad O(t) \in \mathscr{S}^{r-1}(E);$$

a will be called the asymptotic symbol of $A \in \mathscr{A}_0(\not{D})$ *and will be denoted* $\sigma_0(A)$.

(ii) *If* $A, B \in \mathscr{A}_0(\not{D})$, *one has* $\sigma_0(AB) = \sigma_0(A) * \sigma_0(B)$; *here* $*$ *denotes the Getzler multiplication of symbols, defined by the rule*

$$(a * b)(x, \xi) = e^{-\frac{1}{4}R(\frac{\partial}{\partial\xi}, \frac{\partial}{\partial\eta})} a(x, \xi) \wedge b(x, \eta)|_{\xi=\eta},$$

where $R \in \Lambda^2(M) \otimes \Lambda^2(M)$ *is the curvature tensor and the rest of the notation identical to that of* [14].

(3.2) *Examples.* We list below a few symbols which will be needed in the ensuing calculations.

(i) $\sigma_{t^{-1}}(t\not{D})(x, \xi) = it^{-1}\xi$;

(ii) $\sigma_{t^{-1}}(t^2\not{D}^2)(x, \xi) = -|\xi|^2 + Q + \frac{1}{4}t^2\tau(R)$,

where $Q = \nabla_E^2$ is the curvature of (E, ∇_E) and $\tau(R)$ is the scalar curvature of M;

(iii) $\sigma_{t^{-1}}([t\not{D}, f])(x, \xi) = df, \quad f \in \mathscr{A}$

(iv) $\sigma_{t^{-1}}([t^2\not{D}^2, f])(x, \xi) = 2it\langle df, \xi\rangle + t^2\Delta f$;

(v) $\sigma_{t^{-1}}([t^2\not{D}^2, f]t\not{D})(x, \xi) = -2\langle df, \xi\rangle\xi + O(t), O(t) \in \mathscr{S}^2(E)$;

(vi) $\sigma_{t^{-1}}((\lambda + t^2\not{D}^2)^{-1})(x, \xi) = (\lambda - H(x, \xi))^{-1} + O(t), O(t) \in \mathscr{S}^{-3}(E)$,

where

$$H(x, \xi) = |\xi|^2 - \frac{1}{2}R\left(\xi, \frac{\partial}{\partial\xi}\right) - \frac{1}{16}R \wedge R\left(\frac{\partial}{\partial\xi}, \frac{\partial}{\partial\xi}\right) - Q$$

(the notational conventions being the same as in [14]);

(vii) $\sigma_{t^{-1}}(u(-t^2\not{D}^2))(x, \xi) = u(H)(x, \xi) + O(t), O(t) \in \mathscr{S}^{-\infty}(E)$, for any $u(-\not{D}^2) \in \mathscr{A}(\not{D})$.

The first five formulae are easy to check, the sixth is proved in [14, p. 177] and so is (vii) for the special case $u(x) = e^{-x}$. To prove (vii) in the general case, one writes $u = v^{(N)}$, with v Schwartz, and then one employs a similar argument as for the exponential function. □

We now return to the operator Π. Denoting $u(x) = e^{-\frac{1}{2}x}w(x)$, $x \geq 0$, $A = u(-\not{D}^2)\not{D}$, $A^j = u(-\not{D}^2)[\not{D}, f^j]$ and $B^j = [u(-\not{D}^2), f^j]\not{D}$, for $j = 0, \ldots, 2q$, we can write

$$\Pi = (e^{\not{D}^2} - A)f^0([e^{\not{D}^2}, f^1] + A^1 + B^1)([e^{\not{D}^2}, f^2] - A^2 - B^2)\ldots$$
$$\ldots([e^{\not{D}^2}, f^{2q-1}] + A^{2q-1} + B^{2q-1})([e^{\not{D}^2}, f^{2q}] - A^{2q} - B^{2q}).$$

Of course, we are only interested in the "diagonal part" Π^{ev}, more precisely in estimating $Tr_s\Pi(t) = Tr_s\Pi^{ev}(t)$ near $t = 0$. With this goal in mind, we shall make, in the course of the proof of the following lemma, a preliminary assessment of the terms involved in the expression of Π^{ev} (after the multiplications have been performed).

(3.3) LEMMA. Π^{ev} *has asymptotic order* 0.

Proof. As a general remark, we note that since A, A^j, B^j are odd, only the products containing an even number of them will occur in the expression of Π^{ev}. Also, except for A which has asymptotic order 1, all other factors have asymptotic order 0.

We now take an inventory of the terms involved, beginning with those which will give no contribution to the asymptotic symbol.

1°. *Terms starting with e^{D^2} and containing at least one factor of the form* $[e^{D^2}, f^j]$. Clearly, $\sigma_0([e^{D^2}, f^j]) = 0$. Since the rest of the factors have asymptotic order 0, the asymptotic symbol of such a term is 0.

2°. *Terms starting with e^{D^2} and containing at least two factors B^j, B^k.* They have asymptotic order 0, and we can actually see that the asymptotic symbol of such a term vanishes. Indeed, remark that:

$$[D, u(-D^2)] = 0,$$
$$[D, [u(-D^2), f]) = [u(-D^2), [D, f]] \in \mathscr{A}_{-1}(D),$$

and the graded commutator

$$[D, [D, f]]_+ = [D^2, f] \in \mathscr{A}_{-1}(D).$$

Thus, without changing the asymptotic symbol of the term, we can bring the two D's next to each other, effectively lowering the asymptotic order by 2.

3°. *Terms starting with A and containing more than one factor of the form* $[e^{D^2}, f^j]$ (hence at least three such factors). They manifestly have asymptotic symbol 0.

4°. *Terms starting with A and containing at least a factor B^j.* Note that they have asymptotic order 0, since they must contain a factor $[e^{D^2}, f^k]$. In fact, arguing as in case 2°, one sees that their asymptotic symbol vanishes.

Here are the terms which will contribute to the asymptotic symbol:

5°. $T = e^{D^2} f^0 A^1 \gamma A^2 \gamma \ldots A^{2q} \gamma = (-1)^q e^{D^2} f^0 A^1 \ldots A^{2q}$; it clearly has asymptotic order 0.

6°. $T_j = e^{D^2} f^0 A^1 \gamma \ \ldots \ A^{j-1} \gamma B^j \gamma A^{j+1} \gamma \ \ldots \ A^{2q} \gamma = (-1)^q e^{D^2} f^0 A^1 \ \ldots \ A^{j-1} B^j A^{j+1} \ldots A^{2q}$, $j = 1, \ldots, 2q$; again, all factors have asymptotic order 0.

7°. $Z_j = -A f^0 A^1 \gamma \ \ldots \ A^{j-1} \gamma [e^{D^2}, f^j] \gamma A^{j+1} \gamma \ \ldots \ A^{2q} \gamma = (-1)^{q+j} A f^0 A^1 \ldots A^{j-1} [e^{D^2}, f^j] A^{j+1} \ldots A^{2q}$, $j = 1, \ldots, 2q$; the fact that A has asymptotic order 1 is compensated by the presence of $[e^{D^2}, f^j]$. □

We are now ready to begin the actual calculations.

(3.4) *Computation of the contribution of T.*

One has

$$\begin{aligned}\sigma_0(T) &= (-1)^q \sigma_0(e^{D^2}) * f^0 * \sigma_0(u(-D^2)) * df^1 * \ldots * \sigma_0(u(-D^2)) * df^{2q} \\ &= (-1)^q \sigma_0(e^{D^2} u(-D^2)^{2q}) \wedge f^0 df^1 \wedge \ldots \wedge df^{2q}.\end{aligned}$$

But

$$e^{-x} u(x)^{2q} = e^{-x} \left(\frac{e^{-x} - e^{-2x}}{x} \right)^q = \int_1^2 \ldots \int_1^2 e^{-(1 + s_1 + \ldots + s_q)x} ds_1 \ldots ds_q$$

hence

$$\sigma_0(e^{D^2} u(-D^2)^{2q}) = \int_1^2 \ldots \int_1^2 \sigma_0(e^{-(1 + s_1 + \ldots + s_q) D^2}) ds_1 \ldots ds_q.$$

By (3.2) (vii),

$$\sigma_0(e^{s\not{D}^2})(x,\xi) = e^{-sH}(x,\xi),$$

which really means $(e^{-sH}\cdot 1)(x,\xi)$. Since $R\left(\xi,\frac{\partial}{\partial\xi}\right)$ commutes with the other terms in $H(x,\xi)$, and so does Q, one obtains

$$\sigma_0(e^{s\not{D}^2})(x,\xi) = e^{-s(|\xi|^2 - \frac{1}{16}R(\frac{\partial}{\partial\xi},\frac{\partial}{\partial\xi}))}e^{sQ}.$$

The exponent of the first exponential is the harmonic oscillator H_R, over the ring $\Lambda T_x^* M$, with $\frac{1}{\textit{frequency}} = \frac{1}{16} R \wedge R$. Its Fourier transform is therefore the harmonic oscillator

$$K_R = -\Delta + \tfrac{1}{16} R \wedge R(X,X), \quad X \in T_x M \cong \mathbb{R}^{2n}.$$

By Mehler's formula (see [14, p. 173] or [15, p. 114]), its heat kernel has the expansion

$$\begin{aligned} e^{-sK_R}(X,Y) = {} & (4\pi s)^{-n} \det\left(\frac{sR/2}{\sinh sR/2}\right)^{1/2} \\ & \times \exp\sum_{i,j} -\frac{1}{4s}\left[\left(\frac{sR/2}{\tanh sR/2}\right)_{ij}(X^iX^j + Y^iY^j)\right. \\ & \left. - 2\left(\frac{sR/2}{\sinh sR/2}\right)_{ij} X^iY^j\right]. \end{aligned}$$

In particular,

$$\begin{aligned} \int (e^{-sH_R}\cdot 1)(\xi)\,d\xi = \int e^{-sH_R}(\xi,\eta)\,d\xi\,d\eta &= (2\pi)^{2n} e^{sK_R}(0,0) \\ &= \pi^n s^{-n} \det\left(\frac{sR/2}{\sinh sR/2}\right)^{1/2}. \end{aligned}$$

Therefore, setting $s = 1 + s_1 + \ldots + s_q$, one has

$$\begin{aligned} \int \sigma_0(e^{\not{D}^2} u(-\not{D}^2)^{2q})(x,\xi)\,d\xi = \pi^n \int_1^2 \ldots \int_1^2 s^{-n} \det\left(\frac{sR/2}{\sinh sR/2}\right)^{1/2} \\ \wedge e^{sQ}\, ds_1 \ldots ds_q. \end{aligned}$$

Applying the supertrace, which amounts to multiplying by $\left(\frac{2}{i}\right)^n$ and taking the top degree component, one finally obtains:

$$\begin{aligned} (2\pi)^{-2n} \int_{T^*M} tr_s \sigma_0(T)\,dx\,d\xi = \beta_q(-1)^q (2\pi i)^{-n} \int_M \det\left(\frac{R/2}{\sinh R/2}\right)^{1/2} \\ \wedge\, tr\, e^Q \wedge f^0 df^1 \wedge \ldots \wedge df^{2q}, \end{aligned}$$

where

$$\beta_q = \int_1^2 \ldots \int_1^2 (1 + s_1 + \ldots + s_q)^{-q}\, ds_1 \ldots ds_q.$$

(3.5) *Computation of the contribution of* $\sum_{j=1}^{2q} T_j$.

One has

$$T_j = (-1)^q e^{\not{D}^2} f^0 A^1 \dots A^{j-1} \left(\frac{1}{2\pi i} \int_C u(\lambda)[(\lambda + \not{D}^2)^{-1}, f^j] \, d\lambda \right) \not{D} A^{j+1} \dots A^{2q}$$

where C is a suitable contour in $\mathbb{C}$, oriented counterclockwise. Since

$$[(\lambda + \not{D}^2)^{-1}, f^j] = -(\lambda + \not{D}^2)^{-1} [\not{D}^2, f^j] (\lambda + \not{D}^2)^{-1},$$

one obtains

$$T_j = (-1)^{q+1} e^{\not{D}^2} f^0 A^1 \dots A^{j-1} \left(\frac{1}{2\pi i} \int_C u(\lambda)(\lambda + \not{D}^2)^{-1} [\not{D}^2, f^j] \right.$$
$$\left. \times \not{D}(\lambda + \not{D}^2)^{-1} \, d\lambda \right) A^{j+1} \dots A^{2q}.$$

At this point we take advantage of the fact that only the (super) trace is needed and replace T_j by

$$\tilde{T}_j = \frac{(-1)^{q+1}}{2\pi i} \int_C (-1)^{2q+j} (\lambda + \not{D}^2)^{-1} A^{j+1} \dots A^{2q} e^{\not{D}^2} f^0 A^1$$
$$\dots A^{j-1} u(\lambda)(\lambda + \not{D}^2)^{-1} [\not{D}^2, f^j] \not{D} \, d\lambda,$$

thus bringing the two resolvents on the same side of $[\not{D}^2, f^j]$. Clearly,

$$\lim_{t \to 0} Tr_s T_j(t) = \lim_{t \to 0} Tr_s \tilde{T}_j(t) = (2\pi)^{-2n} \int_{T^*M} tr_s \sigma_0(\tilde{T}_j) \, dx \, d\xi.$$

Now in computing $\sigma_0(\tilde{T}_j)$ we can pass the first $\sigma_0((\lambda + \not{D}^2)^{-1})$ over the intermediate terms all the way till it reaches the second $\sigma_0((\lambda + \not{D}^2)^{-1})$. Thus, we get:

$$\sigma_0(\tilde{T}_j) = (-1)^{q+1+2q-j} df^{j+1} \wedge \dots \wedge df^{2q} \wedge f^0 df^1 \wedge \dots \wedge df^{j-1}$$
$$* \sigma_0(e^{\not{D}^2} u^{2q-1}(-\not{D}^2) u'(-\not{D}^2)) * \sigma_0([\not{D}^2, f^j] \not{D}).$$

By (3.2) (v), $\sigma_0([\not{D}^2, f^j]\not{D})(x, \xi) = -2\langle df^j, \xi \rangle \xi$ and by (3.2) (vii),

$$\sigma_0(e^{\not{D}^2} u^{2q-1}(-\not{D}^2) u'(-\not{D}^2)) = e^{-H} u^{2q-1}(H) u'(H).$$

Introducing these expressions in the above formula and, at the same time rearranging the df's, we obtain:

$$\sigma_0(\tilde{T}_j)(x, \xi) = (-1)^q 2 f^0 df^1 \wedge \dots \wedge df^{j-1} * (e^{-H} u^{2q-1}(H) u'(H))(x, \xi)$$
$$* \langle df^j, \xi \rangle \xi * df^{j+1} \wedge \dots \wedge df^{2q}.$$

As before, we can rewrite the term in H as follows:

$$\begin{aligned}
e^{-H} u^{2q-1}(H) u'(H) &= \frac{1}{2q} e^{-H} \frac{d}{d\lambda} u^{2q} \bigg|_{\lambda = H} \\
&= \frac{1}{2q} e^{-H} \frac{d}{d\lambda} \int_1^2 \dots \int_1^2 e^{-(s_1 + \dots + s_q)\lambda} \, ds_1 \dots ds_q \bigg|_{\lambda = H} \\
&= -\frac{1}{2q} \int_1^2 \dots \int_1^2 (s_1 + \dots + s_q) e^{-(1 + s_1 + \dots + s_q)H} \, ds_1 \dots ds_q \\
&= -\frac{1}{2q} \int_1^2 \dots \int_1^2 (s - 1) e^{-sH} \, ds_1 \dots ds_q,
\end{aligned}$$

where we have set $s = 1 + s_1 + \dots + s_q$.

Thus,

$$\lim_{t \to 0} Tr_s T_j(t) = (2\pi)^{-2n}\left(\frac{2}{i}\right)^n (-1)^q \int_M f_0\, df_1 \wedge \dots \wedge df^{j-1} \wedge \omega^j \wedge df^{j+1} \wedge \dots \wedge df^{2q},$$

where

$$\omega_j = -\frac{1}{q}\int_1^2 \dots \int_1^2 (s-1)\left(\int (e^{-sH}\cdot 1) * \langle df^j, \xi\rangle \xi\, d\xi\right) ds_1 \dots ds_q.$$

Now, for any $f \in C^\infty(M)$,

$$\langle df, \xi\rangle \xi = \sum_{a,b} \xi^a \xi^b e_a(f) e_b^*,$$

where $\{e_a\}$ is an orthonormal basis of $T_x M$ and $\{e_b^*\}$ its dual basis. On the other hand, given any symbol $q(x, \xi)$, of even degree as a form, one checks by a straightforward calculation that

$$\begin{aligned} q(x,\xi) * \xi^a \xi^b = \Big[& \xi^a \xi^b - \frac{1}{4}\xi^a R\left(\frac{\partial}{\partial \xi}, e_b\right) - \frac{1}{4}\xi^b R\left(\frac{\partial}{\partial \xi}, e_a\right) \\ & + \frac{1}{8} R\left(\frac{\partial}{\partial \xi}, e_a\right) \wedge R\left(\frac{\partial}{\partial \xi}, e_b\right)\Big] q(x,\xi), \end{aligned}$$

therefore

$$\begin{aligned} q(x,\xi) * \langle df, \xi\rangle \xi = \Big[& \sum_{a,b} \xi^a \xi^b e_a(f) e_b^* - \frac{1}{4}\sum_{a,b} \xi^a R\left(\frac{\partial}{\partial \xi}, e_b\right) \wedge e_b^* e_a(f) \\ & - \frac{1}{4}\sum_{a,b} \xi^b R\left(\frac{\partial}{\partial \xi}, e_a\right) \wedge e_b^* e_a(f) \\ & + \frac{1}{8}\sum_{a,b} R\left(\frac{\partial}{\partial \xi}, e_a\right) \wedge R\left(\frac{\partial}{\partial \xi}, e_b\right) e_a(f) e_b^* \Big] q(x,\xi) \\ = \Big[& \langle df, \xi\rangle \xi - \frac{1}{4}\langle df, \xi\rangle \sum_b R\left(\frac{\partial}{\partial \xi}, e_b\right) \wedge e_b^* - \frac{1}{4}\xi \wedge R\left(\frac{\partial}{\partial \xi}, \operatorname{grad} f\right) \\ & + \frac{1}{8} R\left(\frac{\partial}{\partial \xi}, \operatorname{grad} f\right) \wedge \sum_b R\left(\frac{\partial}{\partial \xi}, e_b\right) e_b^* \Big] q(x,\xi). \end{aligned}$$

The second and the fourth term vanish since

$$\sum_b R(e_c, e_b) \wedge e_b^* = \sum_{k,l,b} \langle R(e_c, e_b) e_k, e_l\rangle e_k^* \wedge e_l^* \wedge e_b^* = 0,$$

by the Bianchi identity. It follows that

$$\begin{aligned} \tilde{\omega}_j \overset{\text{def}}{=} \int (e^{-sH}\cdot 1)(\xi) * \langle df^j, \xi\rangle \xi\, d\xi = & \int \langle df, \xi\rangle \xi \wedge (e^{-sH}\cdot 1)(\xi)\, d\xi \\ & - \frac{1}{4}\int \xi \wedge R\left(\frac{\partial}{\partial \xi}, \operatorname{grad} f\right)(e^{-sH}\cdot 1)(\xi)\, d\xi. \end{aligned}$$

Let us prove that the second integral vanishes. One first remarks that

$$R\left(\frac{\partial}{\partial \xi}, \operatorname{grad} f\right)\xi = \sum_b R(e_b, \operatorname{grad} f) e_b^* = 0,$$

as above. Therefore,

$$\int \xi \wedge R\left(\frac{\partial}{\partial \xi}, \operatorname{grad} f\right)(e^{-sH}\cdot 1)(\xi)\,d\xi = \int R\left(\frac{\partial}{\partial \xi}, \operatorname{grad} f\right)(\xi \wedge (e^{-sH}\cdot 1)(\xi))\,d\xi = 0,$$

since $\xi \wedge (e^{-sH}\cdot 1)(\xi)$ is a Schwartz function (with values in $\Lambda T_x^* M$).

Thus,

$$\begin{aligned}\tilde{\omega}_j &= \int \langle df^j, \xi\rangle \xi \wedge (e^{-sH}\cdot 1)(\xi)\,d\xi \\ &= \sum_{a,b} e_a(f^j) e_b^* \wedge e^{sQ} \int \xi^a \xi^b (e^{-sH}\cdot 1)(\xi)\,d\xi.\end{aligned}$$

We now use the harmonic oscillator to evaluate the latter integrals. Namely, by Fourier transforming,

$$\begin{aligned}\mathscr{I}_{ab} &\overset{\text{def}}{=} \int \xi^a \xi^b (e^{-sH_R}.\, 1)(\xi)\,d\xi = \int \xi^a \xi^b e^{-sH_R}(\xi, \eta)\,d\xi\,d\eta \\ &= -(2\pi)^{2n}\left(\frac{\partial}{\partial X^a}\frac{\partial}{\partial X^b}\, e^{-sK_R}\right)(0,0),\end{aligned}$$

which can be computed from Mehler's formula, giving

$$\mathscr{I}_{ab} = \frac{1}{2}\pi^n s^{-(n+1)}\left(\frac{sR/2}{\tanh sR/2}\right)_{ab} \det\left(\frac{sR/2}{\tanh sR/2}\right)^{1/2}.$$

It follows that

$$\tilde{\omega}_j = \frac{1}{2}\pi^n s^{-(n+1)} e^{sQ} \det\left(\frac{sR/2}{\sinh sR/2}\right)^{1/2} \sum_{a,b} e_a(f^j) e_b^* \wedge \left(\frac{sR/2}{\tanh sR/2}\right)_{ab}.$$

But, again because of the Bianchi identity,

$$\sum_b e_b^* \wedge \left(\frac{sR/2}{\tanh sR/2}\right)_{ab} = \sum_b e_b^* \delta_{ab} = e_a^*$$

which implies that

$$\tilde{\omega}_j = \frac{1}{2}\pi^n s^{-(n+1)} e^{sQ} \det\left(\frac{sR/2}{\sinh sR/2}\right)^{1/2} \wedge df^j$$

and hence

$$\omega_j = -\frac{1}{2q}\pi^n \mathrm{d}f^j \wedge \int_1^2 \cdots \int_1^2 \det\left(\frac{sR/2}{\sinh sR/2}\right)^{1/2} e^{sQ}(s-1)s^{-(n+1)}\,ds_1 \ldots ds_q.$$

It follows that

$$\begin{aligned}\lim_{t\to 0} Tr_s T_j(t) = -\frac{1}{2q}\delta_q(-1)^q(2\pi i)^{-n} \int_M \det\left(\frac{sR/2}{\sinh sR/2}\right)^{1/2} \\ \wedge\, tr e^Q \wedge f^0 df^1 \wedge \ldots \wedge df^{2q}\end{aligned}$$

where

$$\delta_q = \int_1^2 \cdots \int_1^2 (s-1)s^{-(q+1)}\,ds_1 \ldots ds_q = \beta_q - \alpha_q;$$

β_q is the same constant as in (3.4) and

$$\alpha_q = \int_1^2 \dots \int_1^2 s^{-(q+1)} ds_1 \dots ds_q = \frac{q!}{(2q+1)!}.$$

Therefore, the total contribution of the terms T and T_j, $1 \le j \le 2q$, is:

$$\lim_{t\to 0} Tr_s\left(T(t) + \sum_{j=1}^{2a} T_j(t)\right) = (-1)^q (2\pi i)^{-n} \frac{q!}{(2q+1)!} \int_M \det\left(\frac{sR/2}{\sinh sR/2}\right)^{1/2}$$
$$\wedge\, tr e^Q \wedge f^0 df^1 \wedge \dots \wedge df^{2q} \qquad \square$$

(3.6) *Computation of the contribution of* $\sum_{j=1}^{2q} Z_j$.

One has

$$Z_j = (-1)^{q+j} A f^0 A^1 \dots A^{j-1} [e^{D\!\!\!/^2}, f^j] A^{j+1} \dots A^{2q}$$
$$= (-1)^{q+j+1} A f^0 A^1 \dots A^{j-1}$$
$$\times \left(\frac{1}{2\pi i} \int_C e^{-\lambda} (\lambda + D\!\!\!/^2)^{-1} [D\!\!\!/^2, f^j] (\lambda + D\!\!\!/^2)^{-1} d\lambda\right) A^{j+1} \dots A^{2q}.$$

Arguing as in (3.5), we replace it by

$$\tilde{Z}_j = \frac{(-1)^{q+j+1}}{2\pi i}$$
$$\int_C (-1)^{2q-j} (\lambda + D\!\!\!/^2)^{-1} A^{j+1} \dots A^{2q} A f^0 A^1 \dots A^{j-1} e^{-\lambda} (\lambda + D\!\!\!/^2)^{-1} [D\!\!\!/^2, f^j]\, d\lambda,$$

whose asymptotic symbol is

$$\sigma_0(\tilde{Z}_j) = (-1)^{q+j} df^{j+1} \wedge \dots \wedge df^{2q} \wedge f^0 df^1 \wedge \dots \wedge df^{j-1} \sigma_0(-u^{2q}(-D\!\!\!/^2) e^{D\!\!\!/^2})$$
$$* \sigma_0(D\!\!\!/ [D\!\!\!/^2, f^j])$$
$$= 2(-1)^q f^0 df^1 \wedge \dots \wedge df^{j-1} \wedge (u^{2q}(H) e^{-H})(x, \xi) * (df^j, \xi) \xi \wedge df^{j+1} \wedge \dots \wedge df^{2q}.$$

Now, as we have seen in (3.4),

$$u^{2q}(H) e^{-H} = \int_1^2 \dots \int_1^2 e^{-sH} ds_1 \dots ds_q, \quad s = 1 + s_1 + \dots + s_q.$$

Thus,

$$\int \sigma_0(\tilde{Z}_j)(x, \xi) d\xi = 2(-1)^q \int_1^2 \dots$$
$$\int_1^2 f^0 df^1 \wedge \dots \wedge df^{j-1} \wedge \tilde{\omega}_j \wedge df^{j-1} \wedge \dots \wedge df^{2q} ds_1 \dots ds_q.$$

Inserting the expression of $\tilde{\omega}_j$ found in (3.5) we obtain

$$\int \sigma_0(\tilde{Z}_j)(x, \xi) d\xi = (-1)^q \pi^n \int_1^2 \dots \int_1^2 s^{-(n+1)} e^{sQ}$$
$$\wedge \det\left(\frac{sR/2}{\sinh sR/2}\right)^{1/2} ds_1 \dots ds_q \wedge f^0 df^1 \wedge \dots \wedge df^{2q}$$

which in turn gives

$$\lim_{t\to 0} Tr_s Z_j(t) = (-1)^q (2\pi i)^{-n} \alpha_q \int_M \det\left(\frac{sR/2}{\sinh R/2}\right)^{1/2} \wedge tr e^Q \wedge f^0 df^1 \wedge \dots \wedge df^{2q}.$$

Recalling that $\alpha_q = \dfrac{q!}{(2q+1)!}$ it follows that

$$\lim_{t\to 0} Tr_s \sum_{j=1}^{2q} Z_j(t) = 2q(-1)^q (2\pi i)^{-n} \frac{q!}{(2q+1)!} \int_M \det\left(\frac{sR/2}{\sinh R/2}\right)^{1/2} \wedge tr e^Q \wedge f^0 df^1 \wedge \dots \wedge df^{2q}.$$

Before collecting the above results in a single statement, les us recall that the $\hat{A}$-form of a Riemannian manifold with curvature R is defined by

$$\hat{A}(R) = \det\left(\frac{\frac{1}{2\pi i} R/2}{\sinh\left(\frac{1}{2\pi i} R/2\right)}\right)^{1/2}$$

and the Chern character form of a bundle E with connection ∇_E has the expression

$$ch(\nabla_E^2) = tr\, e^{-\frac{1}{2\pi i}\nabla^2_E}.$$

Adjusting the constants to these definitions, we summarize the preceding computation as follows.

(3.7) Proposition. *Let M be a compact, oriented, spin manifold of dimension $2n$, let D_E be the Dirac operator on M with coefficients in a vector bundle E and let $f^0, \dots, f^{2q} \in \mathcal{A} = C^\infty(M)$. With $W(\not{D}_E)$ being the heat kernel idempotent associated to $\not{D}_E$, one has*

$$\lim_{t\to 0} Tr\left(\left(W(t\not{D}_E) - \frac{1-\gamma}{2}\right) f^0 [W(t\not{D}_E), f^1] \dots [W(t\not{D}_E), f^{2q}]\right) = \frac{(-1)^n}{(2\pi i)^q} \frac{q!}{(2q)!} \int_M \hat{A}(R) \wedge ch(\nabla_E^2) f^0\, df^1 \wedge \dots \wedge df^{2q}.$$

The constant multiplying the above integral looks rather mysterious at this point. The reason why precisely these constants should occur will be made clear in the next section.

(3.8) Remark. *Under the same assumptions as above, one has*

$$\lim_{t\to\infty} Tr(\gamma f^0 [W(t\not{D}_E), f^1] \dots [W(t\not{D}_E), f^{2q}]) = 0.$$

The computation is similar to the one above, only easier. In fact, it parallels the treatment of the terms which start with $e^{\not{D}^2}$, except that the factor $e^{\not{D}^2}$ is omitted. It is easy to see that the effect of this change is to replace the constant β_q in (3.4) by

$$\beta_q' = \int_1^2 \dots \int_1^2 (s_1 + \dots + s_q)^{-q} ds_1 \dots ds_q$$

and δ_q in (3.5) by

$$\delta_q' = \int_1^2 \dots \int_1^2 (s_1 + \dots + s_q)(s_1 + \dots + s_q)^{-(q+1)} ds_1 \dots ds_q = \beta_q'.$$

Thus the constant replacing α_q is

$$\alpha'_q = \beta'_q - \delta'_q = 0,$$

hence the claim. □

We conclude the present section with the statement of the general localized index theorem.

Let M be an arbitrary C^∞ manifold and $B \in \Psi^r(M; E, F)^{-1}$ an elliptic pseudo-differential operator on M. If $b = \sigma_{pr}(B)$, then the triple (b, π^*E, π^*F) defines a class in $K^0(T^*M, T^*M - M)$. We denote by $chb \in H^{ev}(T^*M, T^*M - D^*M)$ its image via the Chern homomorphism. Here D^*M denotes the closed unit disk bundle (with respect to some Riemannian metric) and $H^*(T^*M, T^*M - D^*M)$ can be thought of as the cohomology of the de Rham complex $\Lambda^*(T^*M, T^*M - D^*M)$, consisting of differential forms on T^*M with support in D^*M.

(3.9) THEOREM. *Let B be an elliptic pseudo-differential operator on M and let $[\bar{\varphi}] \in \bar{H}_c^{2q}(M)$. Then*

$$Ind_{[\bar{\varphi}]} B = \frac{1}{(2\pi i)^q} \frac{q!}{(2q)!} (-1)^{\dim M} \langle ch\sigma_{pr}(B)\tau(M)\bar{\lambda}[\bar{\varphi}], [T^*M] \rangle.$$

Here $\tau(M) = Todd(TM \otimes \mathbb{C})$ is the index class of M, $H^*(T^*M, T^*M - D^*M)$ is regarded as a module over $H_c^*(M)$ in the usual way and T^*M is given the symplectic orientation.

Proof. For $q = 0$, this is precisely the cohomological statement of the Atiyah–Singer index theorem [3, III]. We shall therefore assume henceforth that $q > 0$. The theorem can be reduced to Proposition (3.6) in a way which parallels the well-known reduction of the general index theorem to the case of Dirac-type operators (see [2] or [16]). Let us indicate briefly the steps involved.

First, if M is not orientable, let $\tilde{M}$ be its orientable double cover. On lifting B to $\tilde{B}$ and $[\bar{\varphi}]$ to $[\bar{\varphi}]^{\sim}$ on $\tilde{M}$, one obtains $Ind_{[\bar{\varphi}]^{\sim}} \tilde{B} = 2 Ind_{[\bar{\varphi}]} B$. This can be easily seen if we choose the representative φ sufficiently localized around the diagonal in M^{2q+1}. On the other hand, the cohomological formula also gets multiplied by 2. Thus, we may assume M orientable.

Secondly, since φ can be chosen with compact support (see Remark (1.6)) and $Ind_\varphi B = Ind_\varphi b$ (where $b = \sigma_{pr}(B)$), bearing in mind the fact that a lift for $\begin{pmatrix} 0 & b^{-1} \\ b & 0 \end{pmatrix}$ can be constructed in an "almost" local fashion, one sees that $Ind_\varphi B$ remains unchanged if we modify M outside a sufficiently large relatively compact domain, and so does the right hand side of the equation. This allows us to reduce the proof to the case when M is compact.

Thirdly, when M is odd-dimensional, the proof can be reduced to the even-dimensional case as follows. Consider on the circle S^1 the operator $C: C^\infty(S^1) \to C^\infty(S^1)$ defined by

$$C(e^{in\theta}) = \begin{cases} ne^{i(n-1)\theta}, & \text{for } n \geq 0 \\ ne^{in\theta}, & \text{for } n \leq 0. \end{cases}$$

It is elliptic, pseudo-differential, $Index(C) = 1$ and $\langle ch\sigma_{pr}(C), [T^*S^1] \rangle = -1$ (see [16], Lemma 3.9.4]). On $N = M \times S^1$ form the operator

$$D = B \bar{\otimes} C = \begin{pmatrix} 1 \otimes C & B^* \otimes 1 \\ B \otimes 1 & -1 \otimes C^* \end{pmatrix}$$

acting on $(L^2(M; E) \otimes L^2(S^1)) \oplus (L^2(M, F) \otimes L^2(S^1))$. Although not quite a pseudo-differential operator, D can be uniformly approximated by pseudo-differential operators in their natural topology (after adjusting its order to 1, to match the order of C). One easily checks that

$$D^*D = \begin{pmatrix} B^*B \otimes I + I \otimes C^*C & 0 \\ 0 & BB^* \otimes 1 + 1 \otimes CC^* \end{pmatrix}$$

therefore,

$$e^{-D^*D} = \begin{pmatrix} e^{-sB^*B} \otimes e^{-sC^*C} & 0 \\ 0 & e^{-sBB^*} \otimes e^{-sCC^*} \end{pmatrix}$$

and

$$e^{-sD^*D}D^* = \begin{pmatrix} e^{-sB^*B} \otimes e^{sC^*C}C & e^{-sB^*B}B^* \otimes e^{sC^*C} \\ e^{-sBB^*}B \otimes e^{-sCC^*} & -e^{-sBB^*} \otimes e^{-sCC^*}C \end{pmatrix}$$

Consider now the parametrix

$$Q(D) = \frac{1-e^{\frac{1}{2}D^*D}}{D^*D}\, D^* = \int_0^1 e^{-\frac{s}{2}D^*D}D^*\, ds.$$

We "localize" it, but only around the diagonal of M, by means of a cut-off function $\alpha \in C^\infty(M \times M)$ as in the proof of Lemma (2.5), that is, we replace it by the parametrix

$$\tilde{Q}(D) = \int_0^1 \alpha e^{-\frac{s}{2}D^*D}D^*\alpha ds.$$

Assuming, as we may, that B is also sufficiently localized, one has

$$Ind_{\varphi \otimes 1} D = \tau(\varphi \otimes 1)(\tilde{R}(D), \dots, \tilde{R}(D)),$$

with $(\varphi \otimes 1)((x^0, \theta^0), \dots, (x^{2q}, \theta^{2q})) = \varphi(x^0, \dots, x^{2q})$ and $\tilde{R}(D) =$ the "difference idempotent" associated to $\tilde{Q}(D)$.

Remark that if we introduce a "coupling constant" in the above construction, the index will not change. More precisely, setting $D_\beta = D \hat{\otimes} \beta C$, one has

$$Ind_{\varphi \otimes 1} D = Ind_{\varphi \otimes 1} D_\beta = \tau(\varphi \otimes 1)(\tilde{R}(D_\beta), \dots, \tilde{R}(D_\beta)),$$

with $(\varphi \otimes 1)((x^0, \theta^0), \dots, (x^{2q}, \theta^{2q})) = \varphi(x^0, \dots, x^{2q})$ and $\tilde{R}(D) =$ the "difference idempotent" associated to $\tilde{Q}(D)$.

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$$Ind_{\varphi \otimes 1} D = Ind_{\varphi \otimes 1} D_\beta = \tau(\varphi \otimes 1)(\tilde{R}(D_\beta), \dots, \tilde{R}(D_\beta)).$$

We can now let $\beta \to \infty$, which has the effect of "decoupling" the two cocycles in the right hand side of the above equation. Indeed, since $e^{-\beta^2 C^*C}$ converges strongly to the projection $|1\rangle$ onto $Ker C = \mathbb{C} \cdot 1$ and $e^{-\beta^2 CC^*}$ to 0, one has $e^{-s\beta^2 D^*D}D^* \to e^{-sB^*B}B^* \otimes |1\rangle$ strongly, therefore (s)- $\lim_{\beta \to \infty} \tilde{R}(D_\beta) = \tilde{R}(B)$. This, in turn, implies that

$$\lim_{\beta \to \infty} \tau(\varphi \otimes 1)(\tilde{R}(D), \dots, \tilde{R}(D)) = \tau(\varphi)(\tilde{R}(0), \dots, \tilde{R}(0)),$$

therefore

$$Ind_{\varphi \otimes 1} D = Ind_\varphi B.$$

On the other hand, it follows from [16; Lemma 3.9.3] that

$$\langle ch\sigma_{pr}(D)\cdot\tau(N)\cdot\bar{\lambda}[\bar{\varphi}\otimes 1],[T^*N]\rangle = \langle ch\sigma_{pr}(B)\cdot\tau(M)\cdot\bar{\lambda}[\bar{\varphi}\otimes 1],(T^*M)\rangle \cdot\langle ch\sigma_{pr}(C),[T^*S^1]\rangle = -\langle ch\sigma_{pr}(B)\cdot\tau(M)\cdot\bar{\lambda}[\bar{\varphi}],[T^*M]\rangle.$$

This completes the third reduction step.

Thus, we are reduced to proving the theorem for an elliptic symbol on an even-dimensional, oriented, Riemannian manifold. Since modulo 2-torsion, $K_c^0(T^*M)$ is spanned by symbols of generalized signature operators, it is enough to check the statement for such operators. But a generalized signature operator *is* a generalized Dirac operator (in the sense of [19]). It remains to remark that the proof of Proposition (3.7), based on symbolic calculus, has a local nature and therefore applies not only to Dirac operators with twisted coefficients but to generalized Dirac operators as well. □

§4. ASYMPTOTIC CHERN CHARACTER IN K-HOMOLOGY

In this section we digress to relate the localized analytical indices to the Chern character in K-homology. The link is established via the notion of 'asymptotic Chern character" of a θ-summable Fredholm module, which is interesting in its own right and will be defined below in full generality.

We recall [9] that an *even θ-summable Fredholm module* $(\mathcal{H}, D, \gamma)$ over a unital C^*-algebra A consists of a $\mathbb{Z}_2$-graded Hilbert space $\mathcal{H}$ with grading γ, on which A acts (via a $*$-representation) by even operators, and an unbounded odd selfadjoint operator D such that:

(α) the set $\{a\in A;\, [D,a]$ is densely defined and extends to a bounded operator$\}$ is norm dense in A;

(β) $Tr(e^{-tD^2})<\infty$, $\forall t>0$.

(4.1) LEMMA. *Let A be a unital C^*-algebra and $(\mathcal{H}, D, \gamma)$ an even θ-summable Fredholm module over A. Let $R_t=(e^{-t^2D^2}+v(t^2D^2)itD)\gamma$, where $v(x^2)=e^{-\frac{1}{2}x^2}\left(\frac{1-e^{-x^2}}{x^2}\right)^{1/2}$, $P_t=R_t+\frac{1}{2}(1-\gamma)$ and $F_t=2P_t-1$.*

One has:

(i) $P_t=P_t^*=P_t^2$ *and* $F_t^2=1$;

(ii) $[F_t,a]\in\mathcal{L}^1$ *(trace class operators),* $\forall a\in A$;

(iii) $\lim_{t\to 0}\|[F_t,a]\|=0$, $\forall a\in A$.

Proof. The first assertion is immediate (and was in fact checked in Section 3). The second one is also clear, since $R_t\in\mathcal{L}^1$ thanks to the condition (β) above and $[F_t,a]=[R_t,a]$, $\forall a\in A$, because γ commutes with A.

Let us prove (iii). Given $u\in\mathcal{S}(\mathbb{R})$, write

$$u(D)=\frac{1}{2\pi}\int_{-\infty}^{\infty}\hat{u}(s)e^{isD}ds.$$

Since

$$[e^{isD},a]=\int_0^1 e^{isDt}[isD,a]e^{isD(1-t)}dt,$$

for all $a \in A$ such that $[D, a]$ is bounded, one has

$$\| [u(D), a] \| \leq \| [D, a] \| \frac{1}{2\pi} \int_{-\infty}^{\infty} |\hat{u}(x)| |s| \, ds = C \| [D, a] \|.$$

This easily implies

$$\| F_t, a] \| = \| [R_t, a] \| = O(t),$$

for any $a \in A$ as above. The set of such a's being dense in A and $\{F_t\}$ being a bounded family, the same estimate continues to hold for any $a \in A$.

We now recall the definition of the fundamental bicomplex of [8] associated to a unital Banach algebra $\mathscr{A}$. For any $n \in \mathbb{N} = \{0, 1, \dots\}$ $C^n = C^n(\mathscr{A}, \mathscr{A}^*)$ is the linear space of continuous $n+1$ linear forms φ on A, equipped with the norm

$$\| \varphi \| = \sup_{\|a^i\| \leq 1} |\varphi(a^0, \dots, a^n)|, \quad a^i \in A,$$

and $C^{-n-1} = \{0\}$. The becomplex is obtained by setting $C^{p,q} = C^{p-q}$ for any $p, q \in \mathbb{Z}$ and defining two anticommuting differentials d_1 of degree 1 and d_2 of degree -1 as follows:

$$d_1 | C^{p,q} = (p - q + 1)b, \qquad d_2 | C^{p,q} = \frac{1}{p-q} B,$$

where

$$(b\varphi)(a^0, \dots, a^{n+1}) = \sum_{j=0}^{n} (-1)^j \varphi(a^0, \dots, a^j a^{j+1}, \dots, a^{n+1}) + (-1)^{n+1} \varphi(a^{n+1} a^0, \dots, a^n),$$

and

$$B\varphi = AB_0 \varphi,$$

with B_0 and A defined by

$$(B_0 \varphi)(a^0, \dots, a^{n-1}) = \varphi(1, a^0, \dots, a^{n-1}) - (-1)^n \varphi(a^0, \dots, a^{n-1}, 1),$$

$$(A\psi)(a^0, \dots, a^{n-1}) = \sum_{j=0}^{n-1} (-1)^{(n-1)j} \psi(a^j, a^{j+1}, \dots, a^{n-1}, a^0, \dots, a^{j-1}).$$

Since we are interested in C^*-algebras, it is perhaps appropriate to point out that the cyclic cohomology of a *C^*-algebra A*, i.e. the *cohomology of cocycles with finite support* in the above bicomplex, is known in many cases and not particularly interesting. At the opposite end, the cohomology with arbitrary supports is trivial (cf. [8; Lemma 36]) for any Banach algebra. To obtain a more useful cohomology theory one needs to restrict the growth of the cocycles. Such a growth condition was employed in [9] for the definition of the *entire cohomology* of a unital Banach algebra $\mathscr{A}$. In our present context, one would take $\mathscr{A}$ to be the subalgebra $\mathscr{A} = \{a \in A; [D, a] \text{ bounded}\}$.

The growth requirement we shall introduce here, motivated by the following two lemmas, has the distinctive feature of applying to cochains defined on the whole C^*-algebra.

(4.2) LEMMA. *With the notation of Lemma* (4.1), *define for $t > 0$ and $q \in \mathbb{N}$ the cochains $\tau_{2q}^t \in C^{2q}$ as follows*:

$$\tau_0^t(f^0) = Tr(R_t f^0),$$

$$\tau_{2q}^t(f^0, \dots, f^{2q}) = (-1)^q \frac{1.3 \dots (2q-1)}{2^{q+1}} Tr(F_t f^0 [F_t, f^1] \dots [F_t, f^{2q}]), \quad q \geq 1.$$

Given a finite subset Σ of A, let

$$\rho(\tau^t;\Sigma)=\sup_{k\in\mathbb{N}}\overline{\lim_{q\to\infty}}\left(\sup\frac{1}{q!}\,|\tau^t_{2q}(f^0,\ldots,f^{2q})|\right)^{\frac{1}{2q}},$$

where the last sup *is taken over the set* $\{(f^0,\ldots,f^{2q});$ *all* $f^i\in\Sigma$ *except at most* $k\}$. *Then:*

(i) $d_1\tau^t_{2q}+d_2\tau^t_{2q+2}=0\quad\forall t>0,\quad\forall q\in\mathbb{N};$

(ii) $\rho(\tau^t;\Sigma)<\infty$ *and* $\lim_{t\to 0}\rho(\tau^t,\Sigma)=0$.

Proof. (i) is straightforward (see [9]).

(ii) The inequality

$$|Tr(F_tf^0[F_t,f^1]\ldots[F_t,f^{2q}])|\le\|f^0\|\,\|[F_t,f^1]\|_1\sum_{j=2}^{2q}\|[F_t,f^j]\|$$

implies that

$$\rho(\tau^t;\Sigma)\le\sup_{a\in\Sigma}\|[F_t,a]\|.$$

The fact that $\lim_{t\to 0}\rho(\tau^t;\Sigma)=0$ follows now from (4.1) (iii). □

(4.3) Lemma. *The notation being as above, there exists a family of cochains* $\{\sigma^t_{2q+1}\}_{q\in\mathbb{N}}$, $\sigma^t_{2q+1}\in C^{2q+1}$, $\forall t>0$, *such that:*

(i) $d_1\sigma^t_{2q-1}+d_2\sigma^t_{2q+1}=\dfrac{d}{dt}\tau^t_{2q},\forall t>0,\ \forall q\ge 1;$

(ii) $\rho(\sigma^t;\Sigma)<\infty$ *and* $\lim_{t\to 0}\rho(\sigma^t;\Sigma)=0$, $\forall\Sigma\subset A$, Σ *a finite subset.*

Proof. Let $\Delta_t=-\dfrac{d}{dt}F_t$; it is by construction a trace class operator. Define

$$\sigma^t_{2q-1}(f^0,\ldots,f^{2q-1})=\frac{(-1)^q}{2q}\frac{1.3\ldots(2q-1)}{2^{q+1}}\sum_{j=0}^{2q-1}(-1)^j\cdot Tr(F_tf^0[F_t,f^1]\ldots[F_t,f^j]\Delta_t[F_t,f^{j+1}]\ldots[F_t,f^{2q-1}]).$$

The equation (i) is checked by a calculation identical to that in the proof of Prop. 7.3 of [9]. The assertion (ii) is proved by the argument used in Lemma (4.2). □

(4.4) *Definition.* An *(even) asymptotic cocycle* over a C^*-algebra (with unit) A is a family $\{\tau^t_{2q}\}_{q\in\mathbb{N}}$, of cocycles $\tau^t_{2q}\in C^{2q}$ defined for all $t>0$, such that (4.2) (i), (ii) and (4.3) (i), (ii) holds. The particular asymptotic cocycle τ^t defined in Lemma (4.2) will be called the *asymptotic Chern character* of the θ-summable Fredholm module $(\mathscr{H}, D, \gamma)$ and will be denoted $ch^t(\mathscr{H}, D, \gamma)$.

One can develop, following the pattern of [9], a cohomology theory based on asymptotic cocycles. For the purposes of the present paper however, only the pairing with the K-theory is needed.

(4.5) Prosposition. *Let τ^t be an asymptotic cocycle over the C^*-algebra A. Let $e\in M_k(A)$ be an idempotent. If τ^t is normalized (cf. [9]), then:*

(i) *for t sufficiently small the series*

$$\langle \tau^t, e\rangle = \sum_{q=0}^{\infty} \frac{1}{q!}(\tau^t_{2q} \# tr)(e, \ldots, e)$$

is absolutely convergent and its value is independent of t; by definition,

$(\tau^t_{2q} \# tr)(m^0 \otimes f^0, \ldots, m^{2q} \otimes f^{2q}) = tr(m^0 \ldots m^{2q})\tau^t_{2q}(f^0, \ldots, f^{2q})$, *if* $m^j \in M_k(\mathbb{C})$, $f^j \in A$;

(ii) *the above pairing induces an additive map from $K_0(A)$ to $\mathbb{C}$;*

(iii) *if $(\mathscr{H}, D, \gamma)$ is a θ-summable Fredholm module over A, one has for t sufficiently small*

$$\langle ch^t(\mathscr{H}, D, \gamma), [e]\rangle = \operatorname{Index} D_e^+;$$

here $D_e^+ = eDe^+$, with $e^+ = e|e(\mathscr{H}^+ \otimes \mathbb{C}^k)e$.

Proof. (i) By (4.2) (ii), the radius of convergence of the series

$$\sum_{q=0}^{\infty} \frac{1}{q!} z^q(\tau^t_{2q} \# tr)(e, \ldots, e)$$

tends to ∞ as $t \to 0$. In particular, the above series converges absolutely at $z = 1$, for t small enough. Furthermore, since

$$\frac{d}{dt}(\tau^t_{2q} \# tr)(e, \ldots, e) = (d_1\tau^t_{2q-1} \# tr + d_2\tau^t_{2q+1} \# tr)(e, \ldots, e)$$

(cf. (4.3) (i)) and the series corresponding to the right hand side converges absolutely for t small (cf. (4.3) (ii)), it follows that $\frac{d}{dt}\langle \tau^t, e\rangle$ is given by an absolutely convergent series. By [9, Lemma 1.7], this latter series has sum zero.

(ii) After freezing t at a sufficiently small value, one can argue as in [9, Proof of Theorem 1.8] (cf. also the proof of Lemma 2.2).

(iii) The strategy of [9, Section 7], consisting in replacing D, via homotopy, by $D_1 = D - eD(1-e) - (1-e)De$, applies verbatim, giving the claimed equation. $\square$

As mentioned before, the above construction leads naturally to the definition of a cohomology theory for *arbitrary* C^*-algebras, as well as to the construction of a Chern character from K-homology to the cohomology with asymptotic cocycles. This will make the subject of another paper. Here we shall only restate the results of §3 in these terms.

(4.6) THEOREM. *Let M be an even-dimensional, compact, Riemannian manifold, E, a $\mathbb{Z}_2$-graded (complex) Clifford bundle with metric and connection, $\mathscr{H} = L^2(M, E)$ and $\not{D}$ the corresponding (generalized) Dirac operator. Then:*

(i) *$(\mathscr{H}, \not{D}, \gamma)$ is a θ-summable Fredholm module over the C^*-algebra $A = C(M)$ of continuous functions on M;*

(ii) *when restricted to the dense subalgebra $\mathscr{A} = C^1(M)$, the asymptotic character $\tau^t = ch^t(\mathscr{H}, \not{D}, \gamma)$ converges when $t \to 0$ to the cocycle $\tilde{\tau}$ with components*

$$\tilde{\tau}_{2q}(f^0, \ldots, f^{2q}) = \frac{(-1)^q}{(2\pi i)^q}\int_M ch\sigma_{pr}(\not{D}) \wedge \tau(M) \wedge f^0\, df^1 \wedge \ldots \wedge df^{2q},$$

i.e. to the current $ch_*(\not{D})$ which gives the classical Chern character of $[\not{D}] \in K_0(M)$ [5].

Proof. (i) is clear (cf. [9]).

(ii) Replacing F_t by $2P_t - 1 = 2R_t - \gamma$, one has

$$\tau_{2q}^t(f^0, \ldots, f^{2q}) = (-1)^q 2^q 1.3 \ldots (2q-1)(Tr(R_t f^0 [P_t, f^1] \ldots [P_t, f^{2q}])) - \tfrac{1}{2} Tr(\gamma f^0 [P_t, f^1] \ldots [P_t, f^{2q}]).$$

The limit of both terms was computed in Proposition (3.7) and Remark (3.8); although the result was stated only for twisted Dirac operators, it actually holds for generalized Dirac operators, as we already noted in the proof of Theorem (3.9).

The fact that $\sum_{2q > \dim M} \tau_{2q}^t$ converges to 0 is a consequence of the finite summability of the Fredholm module $(\mathscr{H}, D, \gamma)$ and will be proved below in full generality.

(4.7) LEMMA. *Let $(\mathscr{H}, D, \gamma)$ be as in Lemma* (4.1).

(i) *If $f^0, \ldots, f^{2q} \in \mathscr{A} = \{f \in A; [D, f]$ bounded$\}$, then*

$$\frac{1}{q!} |\tau_{2q}^t(f^0, \ldots, f^{2q})| \leq C^q \|f^0\| \prod_{j=1}^{2q} \|[D, f^j]\| \, Tr(e^{-\frac{1}{4}qt^2D^2}) t^2,$$

where C is an absolute constant.

(ii) *If $(\mathscr{H}, D, \gamma)$ is p-summable then $\sum_{2q > p} \tau_{2q}^t$ converges to 0 when $t \to 0$ as a cocycle on $\mathscr{A}$, endowed with the norm $|||a||| = \|a\| + \|[D, a]\|$, $a \in \mathscr{A}$.*

Proof. (i) One has

$$\frac{1}{q!} |\tau_{2q}^t(f^0, \ldots, f^{2q})| = \left| (-1)^q 2^{q-1} \frac{1.3 \ldots (2q-1)}{q!} Tr(F_t f^0 [R_t, f^1] \ldots [R_t, f^{2q}]) \right|$$
$$\leq 2^{2q-1} |Tr(F_t f^0 [R_t, f^1] \ldots [R_t, f^{2q}])|.$$

Recall that $R_t = (e^{-t^2D^2} + e^{-\frac{1}{2}t^2D^2} w(t^2D^2) itD)\gamma$, with $w \in \mathscr{S}(\mathbb{R})$. Arguing as in Lemma (4.1), we can find $c < \infty$, independent of D, such that

$$\max\{\|[e^{-\frac{1}{2}t^2D^2}, f]\|, \|e^{-\frac{1}{2}t^2D^2} w(t^2D^2) D, f]\|\} \leq c \|[D, f]\|, \quad \forall f \in \mathscr{A}.$$

Writing

$$[R_t, f]\gamma = [e^{-\frac{1}{2}t^2D^2}, f] e^{-\frac{1}{2}t^2D^2} + e^{-\frac{1}{2}t^2D^2} [e^{-\frac{1}{2}t^2D^2}, f] + [e^{-\frac{1}{2}t^2D^2}, f] e^{-\frac{1}{2}t^2D^2} w(t^2D^2) itD + e^{-\frac{1}{2}t^2D^2} [e^{-\frac{1}{2}t^2D^2} w(t^2D^2) itD, f],$$

it follows from the above inequality applied to tD that

$$\|[R_t, f]\|_{2q} \leq 4ct \|[D, f]\| \cdot \|e^{-\frac{1}{2}t^2D^2}\|_{2q}.$$

Thus, (i) follows from Hölder's inequality.

(ii) With $k = [p] + 1$, one has

$$\sum_{2q \geq k} Tr(e^{-\frac{1}{4}qt^2D^2}) t^{2q} = t^k Tr(e^{-\frac{1}{4}kt^2D^2}(I - t^2 e^{-\frac{1}{4}kt^2D^2})^{-1}),$$

so that (ii) follows from the estimate (cf. [11]):

$$Tr(e^{-t^2D^2}) = O(t^{-p}), \quad t \to 0. \qquad \square$$

(4.8) *Remark.* Suppose that Proposition (3.7) was established in the less precise form:

$$\lim_{t\to 0} Tr(R_t f^0[P_t, f^1] \dots [P_t, f^{2q}]) = c_q \int_M ch\sigma_{pr}(\not{D}) \wedge \tau(M) \wedge f^0 df^1 \wedge \dots \wedge df^{2q}$$

with the constants c_q undetermined. Then Theorem (4.6) (ii) would have given

$$\tilde{\tau}_{2_q}(f^0, \dots, f^{2q}) = (-1)^q 2^q 1.3 \dots (2q-1) c_q \int_M ch\sigma_{pr}(\not{D}) \wedge \tau(M) \wedge f^0 df^1 \wedge \dots \wedge df^{2q}.$$

In conjunction with (5.4) (iii), this implies

$$\text{Index } \not{D}_E = \sum_{q\geq 0} (-1)^q 2^q \frac{1.3 \dots (2q-1)}{q!} \times c_q \int_M (ch\sigma_{pr}(\not{D}) \wedge \tau(M))_{2(n-q)} \wedge (tr(e^{\nabla_E^2}))_{2q}$$

for any vector bundle with connection (E, ∇_E). Thus, the constant c_q can be determined from the Atiyah–Singer index formula:

$$(-1)^q 2^q \frac{1.3 \dots (2q-1)}{q!} c_q = \frac{(-1)^q}{(2\pi i)^q q!},$$

i.e.

$$c_q = \frac{1}{(2\pi i)^q} \frac{q!}{(2q)!}. \qquad \square$$

§5. HIGHER Γ-INDICES

We shall now extend the Γ-index theorem of Atiyah and Singer ([1, 31]), from the Γ-trace case to higher Γ-cocycles. Throughout this section Γ will denote a countable discrete group, acting properly and freely on a smooth manifold $\tilde{M}$, with compact quotient $M = \Gamma \backslash \tilde{M}$. It will be convenient to fix a Riemannian metric on M end endow $\tilde{M}$ with the lifted metric.

To begin with, let us introduce an algebra which is needed in the construction of the higher Γ-indices. This algebra, to be denoted $\mathscr{A}$, consists of all Γ-invariant, bounded operators A on $L^2(\tilde{M})$ whose Schwartz kernel $A(\tilde{x}, \tilde{y})$, *a priori* only a distribution on $\tilde{M} \times \tilde{M}$ satisfying the Γ-invariance property

$$A(g \cdot \tilde{x}, g \cdot \tilde{y}) = A(\tilde{x}, \tilde{y}), \quad \forall \gamma \in \Gamma,$$

is actually a C^∞ function with compact support modulo Γ. A more complete (and suggestive) notation for this algebra is $C_c^\infty(\tilde{M} \times_\Gamma \tilde{M})$.

We now proceed to construct certain homomorphisms from $\mathscr{A}$ to $\mathbb{C}\Gamma \otimes \mathscr{R}_M$ and $\mathbb{C}\Gamma \otimes \mathscr{L}_M^2$ where $\mathscr{R}_M$ (resp. $\mathscr{L}_M^2$) is the algebra of smoothing operators in $L^2(M)$ (resp. Hilbert Schmidt operators).

Let $\{B_1, \dots, B_r\}$ be an open covering of M by small balls B_i, domains of smooth cross-sections $\beta_i: B_i \to \tilde{M}$ for the canonical projection $\pi: \tilde{M} \to M$. Let $(\chi_i)_{i=1,\dots,r}$ be a smooth partition of unity subordinate to the above covering. We shall assume, as we may, that each $\chi_i^{1/2}$ is a smooth function. The following formula defines a Γ-equivariant isometry U from $L^2(\tilde{M})$ into $L^2(M \times \{1, \dots, r\} \times \Gamma)$:

$$(U\xi)(x, i, g) = \chi_i(x)^{1/2} \xi(g\beta_i(x)) \quad \forall g \in \Gamma,\ x \in M,\ i \in \{1, \dots, r\}.$$

The adjoint of this isometry is given by the following equality:

$$(U^*\eta)(\tilde{x}) = \sum_j \chi_j(x)^{1/2}\eta(x, j, \tilde{x}/\beta_j(x)) \quad \forall \tilde{x} \in \tilde{M},$$

where $\tilde{x}/\beta_j(x) \in \Gamma$ is the unique element g of Γ such that $g\beta_j(x) = \tilde{x}$. For any $A \in \mathscr{A} = C_c^\infty(\tilde{M} \times_\Gamma \tilde{M})$, let $\theta(A) = UAU^*$; its Schwartz kernel is given by:

$$\theta(A)(x, i, g; y, j, h) = \chi_i(x)^{1/2}\chi_j(y)^{1/2}A(g\beta_i(x), h\beta_j(y)).$$

A straightforward calculation shows that

$$\theta(A) = \sum_{g \in \Gamma} \rho(g) \otimes \theta_g(A),$$

where ρ denotes the right regular representation of Γ and $\theta_g(A) \in M_r(\mathscr{R}_M)$ is the matrix of smoothing operators given by:

$$(\theta_g(A))_{ij}(x, y) = \chi_i(x)^{1/2}\chi_j(y)^{1/2}A(\beta_i(x), g\beta_j(y)).$$

The compactness of $\overline{\cup \beta_j(B_j)}$ and of the support of A, together with the fact that Γ acts properly on $\tilde{M}$, ensure that $\theta_g(A) = 0$ except for finitely many g's. Thus $\theta(A)$ belongs to the *algebraic* tensor product $\mathbb{C}\Gamma \otimes M_r(\mathscr{R}_M)$ where $\mathbb{C}\Gamma$ denotes the complex representation ring of Γ.

(5.1) LEMMA. (i) *θ is an algebra homomorphism of $\mathscr{A}$ into $\mathbb{C}\Gamma \otimes M_r(\mathscr{R}_M)$.*

(ii) *The induced homomorphism $\theta_*: K_0(\mathscr{A}) \to K_0(\mathbb{C}\Gamma \otimes \mathscr{R}_M)$ is independent of the choice of $\{B_j, \beta_j, \chi_j\}$.*

Proof. (i) Follows from the equality $U^*U = 1$.

(ii) Let $\{B'_j, \beta'_j, \chi'_j\}$ be another set of data as above. A straightforward computation shows that for any $A \in \mathscr{A}$, one has $\theta'(A) = V\theta(A)V^*$, where V is the partial isometry $V = U'U^* = (V_{ij})$, where $V_{ij} = \rho(g_{ij}) \otimes (\chi'_i\chi_j)^{1/2}$, with $g_{ij} \in \Gamma$ being the unique element of Γ such that $\beta'_i(x) = g_{ij}\beta_j(x)$ for $\forall x \in B'_i \cap B_j$, and $(\chi'_i\chi_j)^{1/2}$ is the multiplication operator by the smooth function $\chi'_i(x)^{1/2}\chi_j(x)^{1/2}$ in $L^2(M)$. Thus, V is a multiplier of the algebra $\mathbb{C}\Gamma \otimes M_r(\mathscr{R}_M)$. By a general result of algebraic K theory, the induced K theory maps θ_* and θ'_* of $K_0(\mathscr{A})$ to $K_0(\mathbb{C}\Gamma \otimes \mathscr{R}_M)$ coincide. □

Using an orthonormal basis of eigenfunctions for the Laplacian Δ on M associated to a given Riemannian metric one can identify the algebra $\mathscr{R}_M$ with the algebra $\mathscr{R}$ of matrices $(a_{ij})_{i,j \in \mathbb{N}}$ such that

$$\sup_{i,j \in \mathbb{N}} i^k j^l |a_{ij}| < \infty \qquad \forall k, l \in \mathbb{N}.$$

This identification, together with the above lemma, give rise to canonical homomorphisms

$$\Theta: K_0(\mathscr{A}) \to K_0(\mathbb{C}\Gamma \otimes \mathscr{R}).$$

With minor and obvious modifications, all of the above remains valid if we introduce bundles into the picture. Thus, $L^2(M)$ gets replaced by $L^2(M, E)$, where E is a (Hermitian) vector bundle over M, $L^2(\tilde{M})$ by $L^2(\tilde{M}, \tilde{E})$, where $\tilde{E} = \pi^*E$, and $C_c^\infty(\tilde{M} \times_\Gamma \tilde{M})$ by $C_c^\infty(\tilde{M} \times_\Gamma \tilde{M}, \tilde{E} \otimes \tilde{E}^*)$, consisting of Γ-invariant, compactly supported mod Γ, smooth kernels A such that

$$A(\tilde{x}, \tilde{y}) \in \mathrm{Hom}(E_y, E_x) \cong E_x \otimes E_y^*, \quad \forall (\tilde{x}, \tilde{y}) \in \tilde{M} \times \tilde{M}.$$

Let now $\tilde{D}: C^\infty(\tilde{M}, \tilde{E}^+) \to C^\infty(\tilde{M}, \tilde{E}^-)$ be a Γ-invariant elliptic differential operator. As explained in [1], $\tilde{D}$ is invertible modulo $\mathscr{A} = C_c^\infty(\tilde{M} \times_\Gamma \tilde{M}, \tilde{E} \otimes \tilde{E}^*)$. More explicitly, one can lift almost local parametrices Q of D to Γ-invariant parametrices $\tilde{Q}$ and $\tilde{D}$, so that $\tilde{S}_0 = I - \tilde{Q}\tilde{D}$, $\tilde{S}_1 = I - \tilde{D}\tilde{Q} \in \mathscr{A}$. From such a parametrix $\tilde{Q}$ one can manufacture the idempotent

$$\tilde{P} = \begin{bmatrix} \tilde{S}_0^2 & \tilde{S}_0(I + \tilde{S}_0)\tilde{Q} \\ \tilde{S}_1 \tilde{D} & I - \tilde{S}_1^2 \end{bmatrix} \in M_2(\mathscr{A}).$$

The corresponding reduced class

$$[\tilde{R}] = [\tilde{P}] - \left[\begin{pmatrix} 0 & 0 \\ 0 & I \end{pmatrix}\right] \in K_0(\mathscr{A})$$

can be shown to be independent of the choice of the almost local parametrix Q.

(5.2) *Definition.* The *K-theoretical index* of a Γ-invariant elliptic operator $\tilde{D}$ on $\tilde{M}$ is the class:

$$ind_\Gamma \tilde{D} = \Theta([\tilde{R}]) \in K_0(\mathbb{C}\Gamma \otimes \mathscr{R}).$$

We shall now show that, at the expense of replacing the algebra $\mathscr{R}_M$ by the algebra $\mathscr{L}_M^2$ of all Hilbert Schmidt operators in $L^2(M)$, we can simplify the construction of θ_*, by means of a Borel cross-section β of $\pi: \tilde{M} \to M$. Indeed, let β be a bounded Borel cross-section, i.e., with $\overline{\beta(M)}$ compact. We define a homomorphism $\theta^\beta: \mathscr{A} \to \mathbb{C}\Gamma \otimes \mathscr{L}^2$ by

$$\theta^\beta(A) = \sum_{g \in \Gamma} \rho(g) \otimes \theta_g(A), \quad \theta_g^\beta(A)(x, y) = A(\beta(x), g\beta(y)).$$

Given $\{B_i, \beta_i, \chi_i\}_{i=1,\dots,r}$ as above, the following $r \times 1$ matrix of multipliers of $\mathbb{C}\Gamma \otimes \mathscr{L}^2$ implements the K-theory equivalence of the maps θ_* and θ_*^β: $V = (V_j)_{j=1,\dots,r}$, with $V_j = \Sigma_i \rho(g_{ij}) \otimes 1_{E_i} \chi_j^{1/2}$, where $(E_i)_{i=1,\dots,s}$ is a suitable Borel partition of M and $g_{ij} \in \Gamma$ is such that $\beta(x) = g_{ij}\beta_j(x)$, $\forall x \in E_i \cap B_j$.

The higher Γ-indices of the elliptic operator $\tilde{D}$ will be obtained by pairing $ind_\Gamma \tilde{D}$ with cyclic cocycles on $\mathbb{C}\Gamma \otimes \mathscr{R}$ constructed from group cocycles on Γ. We recall that the graded cohomology group $H^*(\Gamma) = H^*(\Gamma, \mathbb{C})$ of Γ is by definition the graded homology group associated to the complex $\mathscr{C}^*(\Gamma:\Gamma) = \{\mathscr{C}^*(\Gamma;\Gamma), d\}$, whose p-cochains are functions $c: \Gamma^{p+1} \to \mathbb{C}$ satisfying the invariance condition

$$c(g \cdot g_0, \dots, g \cdot g_p) = c(g_0, \dots, g_p), \quad \forall g, g_0, \dots, g_p \in \Gamma,$$

and with coboundary given by the formula

$$(dc)(g_0, \dots, g_{p+1}) = \sum_{i=0}^{p+1} (-1)^i (g_0, \dots, g_{i-1}, g_{i+1}, \dots, g_{p+1}).$$

Since we only deal with real or complex coefficients, the above complex can be replaced by the subcomplex $\mathscr{C}_\alpha^*(\Gamma:\Gamma)$ (cf. [17]), where

$$\mathscr{C}_\alpha^p(\Gamma:\Gamma) = \{c \in \mathscr{C}^p(\Gamma:\Gamma); c(g_{\tau(0)}, \dots, g_{\tau(p)}) = sgn(\tau)c(g_0, \dots, g_p) \quad \forall \tau \in S_{p+1}\},$$

without altering the cohomology.

Let $c \in Z_\alpha^p(\Gamma:\Gamma)$ be a p-cocycle of the latter complex. According to [8], it defines a cyclic p-cocycle $\tau_c \# tr$, or abbreviated τ_c, on $\mathbb{C}\Gamma \otimes \mathscr{R}$, via the formula

$$\tau_c(f^0 \otimes A^0, \dots, f^p \otimes A^p) = tr(A^0 \dots A^p) \sum_{g_0 g_1 \dots g_p = 1} f^0(g_0) f^1(g_1) \dots f^p(g_p)$$
$$\times c(1, g_1, g_1 g_2, \dots, g_1 g_2 \dots g_p),$$

where $f^0, \ldots, f^p \in \mathbb{C}\Gamma$ and $A^0, \ldots, A^p \in \mathcal{R}$. Note that this cocycle extends to $\mathbb{C}\Gamma \otimes \mathcal{L}^2$ if $p \geq 1$. The pairing theory developed in [8] turns this cyclic cocycle into an additive map $[\tau_c]: K_0(\mathbb{C}\Gamma \otimes \mathcal{R}) \to \mathbb{C}$. Explicitly,

$$[\tau_c]([e] - [f]) = \tilde{\tau}_c(e, \ldots, e) - \tilde{\tau}_c(f, \ldots, f);$$

here e, f are idempotent matrices with entries in $(\mathbb{C}\Gamma \otimes \mathcal{R})^\sim$ = the unital algebra obtained by adjoining the identity to $\mathbb{C}\Gamma \otimes \mathcal{R}$, and $\tilde{\tau}_c$ is the canonical extension of τ_c to $(\mathbb{C}\Gamma \otimes \mathcal{R})^\sim$.

By definition, the (c, Γ)-*index* of $\tilde{D}$ is the number

$$Ind_{(c,\Gamma)} \tilde{D} = [\tau_c](ind_\Gamma \tilde{D}).$$

It only depends on the cohomology class $[c] \in H^*(\Gamma)$ and is linear with respect to $[c]$.

For the clarity of the ensuing discussion, it will be appropriate to recall in some detail the canonical identification of $H^*(\Gamma)$ with the cohomology of a $K(\Gamma, 1)$-space. So let X be such a space and let $\tilde{X} \xrightarrow{\pi} X$ be its universal covering. Following [17], we denote by $\mathscr{C}^*(\tilde{X}:\Gamma)$ the complex whose p-cochains are Γ-invariant functions $c: \tilde{X} \xrightarrow{p+1} \mathbb{C}$. There is an obvious homomorphism $V_X: \mathscr{C}^*(\tilde{X}:\Gamma) \to \mathscr{C}^*(\tilde{X})$, where $\mathscr{C}^*(\tilde{X})$ is the singular complex of $\tilde{X}$. Precisely,

$$(V_X c)(\sigma) = c(\sigma(e_0), \ldots, \sigma(e_p)), \qquad \forall \sigma: \Sigma^p \to \tilde{X},$$

with $\{e_0, \ldots, e_p\}$ denoting the vertices for the standard simplex $\Sigma^p \subset \mathbb{R}^{p+1}$. In fact, V_X maps $\mathscr{C}^*(\tilde{X}:\Gamma)$ to the subcomplex $\mathscr{C}^*_\Gamma(\tilde{X})$ of Γ-invariant singular cochains on $\tilde{X}$. Since this latter subcomplex can be canonically identified with $\mathscr{C}^*(X)$, we thus get a natural homomorphism $V_X: \mathscr{C}^*(\tilde{X}:\Gamma) \to \mathscr{C}^*(X)$. As explained in [17, p. 256], due to the fact that $\tilde{X}$ is contractible, the map induced by V_X on homology, $v_X: H_*(\mathscr{C}^*(\tilde{X}:\Gamma)) \to H^*(X)$, is an isomorphism. On the other hand, any Γ-equivariant map $\eta: \tilde{X} \to \Gamma$ induces a homomorphism $\eta^*: H^*(\Gamma) = H_*(\mathscr{C}^*(\Gamma:\Gamma)) \to H_*(\mathscr{C}^*(\tilde{X}:\Gamma))$. This too is an isomorphism (cf. [17, loc. cit.]), which moreover is independent of η. The composition $\iota = v_X \circ \eta^*: H^*(\Gamma) \cong H^*(X)$ provides the canonical identifiation between the two graded cohomology groups.

We now return to our Γ-principal bundle $\tilde{M} \xrightarrow{\pi} M$. Starting with a group cocycle $c \in Z^p(\Gamma:\Gamma)$, we can consruct an Alexander-Spanier p-cocycle $\bar{\varphi}_c$ on M as follows. Fix an arbitrary cross-section $\beta: M \to \tilde{M}$. Let $(x^0, \ldots, x^p) \in M^{p+1}$; if there is a connected, open subset U of M, which admits a *continuous* local cross-section $\beta_U: U \to \tilde{M}$, such that $(x^0, \ldots, x^p) \in U^{p+1}$, we define

$$\varphi_c(x^0, \ldots, x^p) = c(\beta_U(x^0)/\beta(x^0), \ldots, \beta_U(x^p)/\beta(x^p));$$

otherwise, we set $\varphi_c(x^0, \ldots, x^p) = 0$. Since c is Γ-invariant, the cochain φ_c is well-defined. Moreover, since $dc = 0$, one has $\delta\bar{\varphi}_c = 0$. The Alexander–Spanier cocycle thus defined depends on the choice of the global cross-section β. Its cohomology class, however, is independent of β. As a matter of fact, the following strengthened statement holds.

(5.3) LEMMA. *Let $\psi: M \to X$ be a continuous map to a $K(\Gamma, 1)$-space. If $[c] \in H^*(\Gamma)$, then $[\bar{\varphi}_c] \in \bar{H}^*(M)$ corresponds to $\psi^*(\iota[c]) \in H^*(M)$ via the usual identification between the singular and the Alexander–Spanier cohomology.*

Proof. Let $\tilde{M} = \psi^* \tilde{X} \xrightarrow{\pi} M$ be the pull-back of $\tilde{X} \xrightarrow{\pi} X$. Consider as above, but this time for $\tilde{M} \to M$, the complex $\mathscr{C}^*(\tilde{M}:\Gamma)$ of Γ-invariant "straight" cochains on M, together with its natural homomorphism to the singular complex of M, $V_M: \mathscr{C}^*(\tilde{M}:\Gamma) \to C^*(M)$. Evidently, one has

$$V_M \circ \tilde{\psi}^* = \psi^* \circ V_X,$$

where $\tilde{\psi}: \tilde{M} \to \tilde{X}$ is the canonical lift of $\psi: M \to X$. Therefore, in view of the above discussion, if $c \in Z^p_\alpha(\Gamma:\Gamma)$ then

$$\psi^*(\iota[c]) = (v_M \circ \tilde{\psi}^* \circ \eta^*)[c]$$

for any Γ-equivariant map $\eta: \tilde{X} \to \Gamma$. Given a cross-section $\beta: M \to \tilde{M}$, we can certainly choose η so that

$$(\eta \circ \tilde{\psi})(\tilde{x}) = \tilde{x}/\beta(\pi(\tilde{x})), \qquad \forall \tilde{x} \in \tilde{M}.$$

Thus, if U is a connected open subset of M admitting a continuous cross-section $\beta_U: U \to \tilde{M}$, $\sigma: \Sigma^p \to U$ is a singular p-simplex, and $\tilde{\sigma}$ denotes its Γ-invariant lift to $\tilde{M}$, one has

$$\begin{aligned}
\phi_c(\sigma(e_0), \ldots, \sigma(e_p)) &= c(\beta_U(\sigma(e_0))/\beta(\sigma(e_0)), \ldots, \beta_U(\sigma(e_p))/\beta(\sigma(e_p))) \\
&= c(\tilde{\sigma}(e_0)/\beta(\sigma(e_0)), \ldots, (\tilde{\sigma}(e_p)/\beta(\sigma(e_p))) \\
&= c((\eta \circ \tilde{\psi})(\tilde{\sigma}(e_0)), \ldots, (\eta \circ \tilde{\psi})(\tilde{\sigma}(e_p))) \\
&= V_M(\tilde{\psi}^* \eta^* c)(\sigma) = (\psi^* V_x \eta^* c)(\sigma),
\end{aligned}$$

which is precisely our claim. □

We come now to the main result of this section, which provides a cohomological formula for the higher Γ-indices of a Γ-invariant elliptic operator.

(5.4) THEOREM. *Let M be a compact smooth manifold, Γ a countable discrete group, $\tilde{M} \to M$ a Γ-principal bundle over M and $\tilde{D}$ a Γ-invariant elliptic differential operator on $\tilde{M}$. For any group cocycle $c \in Z^{2q}_\alpha(\Gamma:\Gamma)$, one has*

$$Ind_{(c,\Gamma)}\tilde{D} = \frac{(-1)^{\dim M}}{(2\pi i)^q} \frac{q!}{(2q)!} \langle ch\sigma_{pr}(D)\tau(M)\psi^*(\iota[c]), [T^*M]\rangle,$$

where $\psi: M \to B\Gamma$ is the map classifying the covering $\tilde{M} \to M$.

Proof. For $q = 0$, this is the Atiyah-Singer L^2-index theorem for covering spaces. Thus, we may assume that $q > 0$.

Choose a Borel cross-section $\beta: M \to \tilde{M}$, which is bounded and almost everywhere smooth, and use it to form the Alexander–Spanier cocycle $\bar{\varphi}_c \in \bar{Z}^{2q}_\beta(M)$ corresponding to $c \in Z^{2q}_\alpha(\Gamma:\Gamma)$. By Lemma (5.3), $\psi^*\iota[c]$ and $[\bar{\varphi}_c]$ represent the same cohomology class (modulo canonical identifications). Thus, in view of Theorem (3.9), we are reduced to checking the equality

$$Ind_{(c,\Gamma)}\tilde{D} = Ind_{[\bar{\varphi}_c]}D.$$

Let

$$\tilde{R} = \begin{bmatrix} \tilde{S}_0^2 & \tilde{S}_0(I + \tilde{S}_0) \\ \tilde{S}_1\tilde{D} & -\tilde{S}_1^2 \end{bmatrix} \in M_2(\mathscr{A})$$

be the difference idempotent constructed from a parametrix Q of D by the recipe explained before. Let $\theta: \mathscr{A} \to \mathbb{C}\Gamma \otimes \mathscr{L}^2$ be the homomorphism manufactured from the cross-section

β. One has

$$Ind_{(c,\Gamma)}\tilde{D} = \tau_c(\theta(\tilde{R}), \ldots, \theta(\tilde{R}))$$

$$= \sum_{g_0 g_1 \ldots g_p = 1} Tr(\theta_{g_0}(\tilde{R})\theta_{g_1}(\tilde{R}) \ldots \theta_{g_p}(\tilde{R}))c(1, g_1, g_1 g_2, \ldots, g_1 \ldots g_p),$$

with $p = 2q$. It is convenient to change variables as follows:

$$\gamma_1 = g_1, \gamma_2 = g_1 g_2, \ldots, \gamma_p = g_1 \ldots g_p.$$

One obtains

$$Ind_{(c,\Gamma)}\tilde{D} = \sum_{\gamma_1, \ldots, \gamma_p \in \Gamma} c(1, \gamma_1, \ldots, \gamma_p) Tr(\theta_{\gamma_1}(\tilde{R})\theta_{\gamma_1^{-1}\gamma_2}(\tilde{R}) \ldots \theta_{\gamma_{p-1}^{-1}\gamma_p}(\tilde{R})\theta_{\gamma_p^{-1}}(\tilde{R}))$$

$$= \int_{M^{p+1}} \sum_{\gamma_1, \ldots, \gamma_p} c(1, \gamma_1, \ldots, \gamma_p) tr(\tilde{R}(\beta(x_0), \gamma_1\beta(x_1))\tilde{R}(\gamma_1\beta(x_1), \gamma_2\beta(x_2)) \ldots$$

$$\tilde{R}(\gamma_{p-1}\beta(x_{p-1}), \gamma_p\beta(x_p))\tilde{R}(\gamma_p\beta(x_p), \beta(x_0))dx_1 \ldots dx_p.$$

Observe now that if $\beta_U \colon U \to \tilde{M}$ is a smooth local cross-section, there exists precisely one element $(\gamma_1, \ldots, \gamma_p) \in \Gamma^{p+1}$ such that $(\beta(x_0), \gamma_1\beta(x_1), \ldots, \gamma_p\beta(x_p)) \in \beta_U(U)^{p+1}$, and moreover, $c(1, \gamma_1, \ldots, \gamma_p) = \varphi_c(x_0, \ldots x_p)$. From the construction of $\tilde{R}$, which is obtained by lifting a difference idempotent R on M supported around the diagonal in $M \times M$, it follows immediately that

$$Ind_{(c,\Gamma)}\tilde{D} = \int_{M^{p+1}} \varphi_c(x_0, \ldots, x_p) tr(R(x_0, x_1) \ldots R(x_p, x_0))dx_0 \ldots dx_p$$

$$= Ind_{\varphi_c} D. \qquad \square$$

(5.5) *Remark.* As we shall see in the next section, the above theorem plays a crucial role in our approach to the Novikov conjecture. It is thus worthwhile to sketch a different plausible route for proving it, which bypasses the Localized Index Theorem (3.9). Let $\mathscr{V}^{alg}$ denote that flat bundle of left $\mathbb{C}\Gamma$-modules on M induced by the action of Γ on $\mathbb{C}\Gamma$ given by right multiplication. It defines an element $[\mathscr{V}^{alg}] \in K_0(\mathbb{C}\Gamma \otimes C^\infty(M))$, where the tensor product is the algebraic one. Now by [10; Theorem 4.4] (or rather the even-case analogue), for any closed de Rham current C on M, of dimension $2q$, one has

$$\langle \tau_c \# \tilde{C}, [\mathscr{V}^{alg}] \rangle = \langle C, \psi^*(\imath[c]) \rangle. \tag{5.6}$$

where $\tilde{C}$ is the cyclic cocycle on $C^\infty(M)$ associated to the current C and the left hand side is defined by means of the pairing between cyclic cohomology and K-theory introduced in [8]. For $C = ch_*(D)$, the Chern character in K-homology, the right hand side of (5.6) coincides with the right hand side of the Γ-index formula (5.4). Regarding $\tilde{D}$ as the $\mathbb{C}\Gamma$-operator $D_{\mathscr{V}}$ obtained from D by taking coefficients in the flat $\mathbb{C}\Gamma$-bundle $\mathscr{V}^{alg}$, the left hand side of (5.4) should then correspond to the result of pairing τ_c with the index (á la Mishchenko–Fomenko [27]) of $D_{\mathscr{V}}$ in $K_0(\mathbb{C}\Gamma) \otimes \mathbb{Q}$. However, since $K_0(\mathbb{C}\Gamma)$ is in general reduced to $\mathbb{Z}$, this can only be expected to hold after $\mathbb{C}\Gamma$ is replaced by $\mathbb{C}\Gamma \otimes \mathscr{R}$.

§6. APPLICATION TO THE NOVIKOV CONJECTURE

The applications of the Higher Γ-Index Theorem (5.4) to the Novikov conjecture and its circle of ideas depend on the identification of the "C^*-completion" of the Γ-index, attached in the previous section to a Γ-invariant elliptic operator, as a $C_r^*(\Gamma)$-index in the sense of Mishchenko–Fomenko [27]. We begin, therefore, by establishing this fact.

As before, let $\tilde{M} \to M$ be a Γ-principal bundle over the compact manifold M, and let $\tilde{D}: C^\infty(\tilde{M}, \tilde{E}^+) \to C^\infty(\tilde{M}, \tilde{E}^-)$ be a Γ-invariant elliptic differential operator. Letting Γ act by right translations on $C_r^*(\Gamma)$, the reduced C^*-algebra of Γ, we can form the induced flat $C_r^*(\Gamma)$-vector bundle $\mathscr{V} = C_r^*(\Gamma) \times_\Gamma \tilde{M}$ over M. Then $\tilde{D}$, or equivalently $D: C^\infty(M, E^+) \to C^\infty(M, E^-)$, gives rise to an elliptic $C_r^*(\Gamma)$-operator, $\mathscr{D} = I \otimes D: C^\infty(M, \mathscr{V} \otimes E^+) \to C^\infty(M, \mathscr{V} \otimes E^-)$, whose $C_r^*(\Gamma)$-index, $ind_{C_r^*(\Gamma)}\mathscr{D}$, belongs to $K_0(C_r^*(\Gamma))$.

(6.1) LEMMA. *Let $j: \mathbb{C}\Gamma \otimes \mathscr{R} \to C_r^*(\Gamma) \otimes \mathscr{K}$ be the tensor product of the inclusion maps $\mathbb{C}\Gamma \subsetneq C_r^*(\Gamma)$ and $\mathscr{R} \subsetneq \mathscr{K}$ (= the C^*-algebra of compact operators on a separable infinite-dimensional Hilbert space). Denote by $j_K: K_0(\mathbb{C}\Gamma \otimes \mathscr{R}) \to K_0(C_r^*(\Gamma))$ the induced homomorphism on K-theory. Then*

$$j_K(ind_\Gamma \tilde{D}) = ind_{C_r^*(\Gamma)}\mathscr{D}.$$

Proof. Let $L^2(M, \mathscr{V} \otimes E)$ denote the (left) Hilbert $C_r^*(\Gamma)$-module of L^2-sections of the $C_r^*(\Gamma)$-vector bundle $\mathscr{V} \otimes E$. We shall identify $C^\infty(M, \mathscr{V} \otimes E) \subset L^2(M, \mathscr{V} \otimes E)$ with $C^\infty(\tilde{M}, \tilde{\mathbb{C}}_r^*(\Gamma) \otimes \tilde{E})^\Gamma$, the space of Γ-invariant smooth sections of $\tilde{\mathbb{C}}_r^*(\Gamma) \otimes \tilde{E}$, where $\mathbb{C}_r^*(\Gamma)$ is the trivial $C_r^*(\Gamma)$-bundle $C_r^*(\Gamma) \times M$, $\tilde{\mathbb{C}}_r^*(\Gamma) = \pi^*(\mathbb{C}_r^*(\Gamma)) = C_r^*(\Gamma) \times \tilde{M}$, and Γ acts by right translations on $C_r^*(\Gamma)$ and trivially on $\tilde{E}$. With this identification, $\mathscr{D}$ becomes $I \otimes_\Gamma \tilde{D} =$ the restriction of $I \otimes \tilde{D}: C^\infty(\tilde{M}, \tilde{\mathbb{C}}_r^*(\Gamma) \otimes \tilde{E}^+) \to C^\infty(\tilde{M}, \tilde{\mathbb{C}}_r^*(\Gamma) \otimes \tilde{E}^-)$ to $C^\infty(M, \tilde{\mathbb{C}}_r^*(\Gamma) \otimes \tilde{E}^+)^\Gamma$. Let Q be an almost local parametrix for D and let $\tilde{Q}$ denote its canonical lift to $\tilde{M}$. Then $\mathscr{Q} = I \otimes_\Gamma \tilde{Q}$ is a $C_r^*(\Gamma)$-parametrix for $\mathscr{D}$, i.e. $\mathscr{S}_0 = I - \mathscr{Q}\mathscr{D}$ and $\mathscr{S}_1 = I - \mathscr{D}\mathscr{Q}$ are compact $C_r^*(\Gamma)$-operators. One, therefore, has:

$$ind_{C_r^*(\Gamma)}\mathscr{D} = [\mathscr{P}] - \left[\begin{pmatrix} 0 & 0 \\ 0 & I \end{pmatrix}\right],$$

where

$$\mathscr{P} = \begin{pmatrix} \mathscr{S}_0^2 & \mathscr{S}_0(I + \mathscr{S}_0)\mathscr{Q} \\ \mathscr{S}_1\mathscr{D} & I - \mathscr{S}_1^2 \end{pmatrix} = I \otimes_\Gamma \tilde{P}.$$

Let us note that any smoothing $C_r^*(\Gamma)$-operator $\mathscr{S}: C^{-\infty}(\tilde{M}, \tilde{\mathbb{C}}_r^*(\Gamma) \otimes \tilde{E})^\Gamma \to C^\infty(\tilde{M}, \tilde{\mathbb{C}}_r^*(\Gamma) \otimes \tilde{E})^\Gamma$ is given by a kernel section $(\tilde{x}, \tilde{y}) \mapsto \mathscr{S}(\tilde{x}, \tilde{y}) \in C_r^*(\Gamma) \otimes Hom(E_y, E_x)$ (where $C_r^*(\Gamma)$ is identified with $End_{C_r^*(\Gamma)}(C_r^*(\Gamma))$ via the right multiplication), satisfying the Γ-invariance property:

$$\mathscr{S}(g\tilde{x}, g\tilde{y}) = \delta_g * \mathscr{S}(\tilde{x}, \tilde{y}) * \delta_{g^{-1}}, \quad \forall g \in \Gamma;$$

here $\delta_g(\gamma) = 0$ if $\gamma \neq g$ and $\delta_g(g) = 1$.

Let us choose now a Borel, bounded, almost everywhere smooth cross-section β for $\pi: \tilde{M} \to M$. It yields an isomorphism of $C_r^*(\Gamma)$-modules

$$\beta^*: L^2(M, \mathscr{V} \otimes E) \to L^2(M, \mathbb{C}_r^*(\Gamma) \otimes E)$$

as follows: if $u \in C^\infty(\tilde{M}, \tilde{\mathbb{C}}_r^*(\Gamma) \otimes \tilde{E})^\Gamma$ then

$$(\beta^* u)(x) = u(\beta(x)) \in C_r^*(\Gamma) \otimes E_x, \quad \forall x \in M.$$

Given an operator $\mathscr{S}$ on $L^2(M, \mathscr{V} \otimes E)$, we set

$$\tau(\mathscr{S}) = \beta^* \circ \mathscr{S} \circ (\beta^*)^{-1}.$$

If $\mathscr{S}$ is a smoothing operator as above, such that its kernel section is compactly supported modulo Γ, one can compute explicitly $\tau(\mathscr{S})$ as follows. Let $v \in C^\infty(M, \mathscr{V} \otimes E)$ and

$u = (\beta^*)^{-1}v$; then

$$(\tau(\mathscr{S})v)(x)) = (\mathscr{S}u)(\beta(x)) = \int_{\tilde{M}} u(\tilde{y}) * \mathscr{S}(\beta(x), \tilde{y})d\tilde{y}$$

$$= \int_M \sum_{g\in\Gamma} u(g\tilde{y}) * \mathscr{S}(\beta(x), g\tilde{y})dy$$

$$= \int_M \sum_{g\in\Gamma} u(\tilde{y}) * \delta_{g^{-1}} * \mathscr{S}(\beta(x), g\tilde{y})dy$$

$$= \int_M \sum_{g\in\Gamma} v(y) * \delta_{\beta(y)/\tilde{y}} * \delta_{g^{-1}} * \mathscr{S}(\beta(x), g\tilde{y})dy,$$

and after the substitution $g = h\cdot\beta(y)/\tilde{y}$ one obtains

$$(\tau(\mathscr{S})v)(x) = \sum_{h\in\Gamma}\int_M v(y) * \delta_{h^{-1}} * \mathscr{S}(\beta(x), h\beta(y))dy.$$

In particular, if $\mathscr{S} = I \otimes_\Gamma \tilde{S}$, where $\tilde{S}$ is the lift of an almost local smoothing operator $S\colon C^{-\infty}(M, E) \to C^\infty(M, E)$, then its kernel section is of the form

$$\mathscr{S}(\tilde{x}, \tilde{y}) = \delta_e \otimes \tilde{S}(\tilde{x}, \tilde{y}), \quad (\tilde{x}, \tilde{y}) \in \tilde{M} \times \tilde{M}.$$

Therefore,

$$\tau(\mathscr{S}) = \sum_{g\in\Gamma} r(g) \otimes \theta_g(\tilde{S}),$$

where $r(g)$ is the right translation by $g\in\Gamma$ on $C_r^*(\Gamma)$, i.e. $\tau(\mathscr{S})$ corresponds to $\theta(\tilde{S})$. Then $\tau(P)$ corresponds to $\theta(\tilde{P})$, which implies that

$$ind_{C_r^*(\Gamma)}\mathscr{D} = ind_{C_r^*(\Gamma)}\tau(\mathscr{D}) = j_K(ind_\Gamma\tilde{D}). \qquad \square$$

We now proceed to apply the previous results to the problem of the homotopy invariance of higher signatures. So let M be a compact, oriented, smooth manifold. For most of the discussion, we shall also assume that M is even-dimensional. Let $\psi\colon M \to B\Gamma$ be a continuous map to the classifying space of a countable discrete group Γ. We denote by $\tilde{M}^\psi \xrightarrow{\pi} M$ the pull-back, via ψ, of the universal covering $E\Gamma \xrightarrow{\pi} B\Gamma$. By means of Γ-equivariant "algebraic surgery" (cf. [26, 29]), one associates to the Γ-principal bundle $\tilde{M}^\psi \xrightarrow{\pi} M$ an *a priori* homotopy invariant, Mishchenko's *symmetric signature*, $\sigma[M, \psi] \in L^0(\mathbb{C}\Gamma)$ = the Witt group of (equivalence classes of) non-singular, Hermitian forms over the ring with involution $\mathbb{C}\Gamma$. The inclusion $j\colon \mathbb{C}\Gamma \subsetneq C_r^*(\Gamma)$ induces a homomorphism $j_L\colon L^0(\mathbb{C}\Gamma) \to K_0(C_r^*(\Gamma))$, since $L^0(C_r^*(\Gamma))$ can be naturally identified with $K_0(C_r^*(\Gamma))$.

Now let $D = d + d^*\colon \Lambda^+(M) \to \Lambda^-(M)$ be the signature operator on M^{2n}, let $\tilde{D}^\psi$ be its Γ-equivariant lift to $\tilde{M}^\psi$ and let $\mathscr{D}^\psi$ be the induced $C_r^*(\Gamma)$-signature operator with coefficients in $\mathscr{V} = C_r^*(\Gamma) \times_\Gamma \tilde{M}$. It is known that (see [24], [25], [26])

$$j_L(\sigma[M, \psi]) = ind_{C_r^*(\Gamma)}\mathscr{D}^\psi.$$

Applying Lemma (6.1), we obtain:

(6.2) LEMMA. *With the above notation, one has*

$$j_K(ind_\Gamma\tilde{D}^\psi) = j_L(\sigma[M, \psi]) \in K_0(C_r^*(\Gamma)).$$

In particular, $j_K(ind_\Gamma \tilde{D}^\psi)$ depends only on the oriented homotopy type of M.

Combining this with Theorem (5.4) gives the following partial solution of Novikov's conjecture.

(6.3) PROPOSITION. *Let $H^*_{ex}(\Gamma, \mathbb{C})$ be the subspace of $H^*(\Gamma, \mathbb{C})$ consisting of all classes $\xi \in H^*(\Gamma, \mathbb{C})$ with the following property: there exists a cocycle $c \in Z^*_\alpha(\Gamma:\Gamma)$, with $[c] = \xi$, such that $[\tau_c]: K_0(\mathbb{C}\Gamma \otimes \mathscr{R}) \to \mathbb{C}$ extends to a map $[\tilde{\tau}_c]: K_0(C^*_r(\Gamma)) \to \mathbb{C}$, i.e. $[\tilde{\tau}_c] \circ j_K = [\tau_c]$. Then the higher signatures*

$$Sign_\xi(M, \psi) = \langle L(M)\psi^*(\imath(\xi)), [M] \rangle, \quad \xi \in H^*_{ex}(\Gamma, \mathbb{C}),$$

are invariants of the oriented homotopy type of M.

Proof. Indeed, if $\xi = [c] \in H^{2q}_{ex}(\Gamma, \mathbb{C})$, by Lemma (6.2) and Theorem (5.4), one has

$$\tilde{\tau}_c(j_L(\sigma[M, \psi])) = Ind_{(c,\Gamma)}\tilde{D}^\psi = \frac{2^n}{(2\pi i)^q}\frac{q!}{(2q)!}\langle \mathscr{L}(M)\psi^*(\imath(\xi)), [M] \rangle,$$

where $\mathscr{L}(M)$ is the stable Hirzebruch class of M (cf. [3, III], p. 577). The left hand side depends only on the homotopy type of M and the right hand side is proportional to $Sign_\xi(M, \psi)$. □

Thus, proving the Novikov conjecture amounts to showing that the "extendable" cohomology exhausts the entire cohomology ring of Γ. We shall see that this is the case when Γ is hyperbolic. But first, we need the following technical lemma.

(6.4) LEMMA. *Let Γ be a finitely generated discrete group, equipped with a word length function $g \mapsto |g|$. Let $\mathscr{C} = \mathscr{C}(\mathbb{C}\Gamma \otimes \mathscr{R})$ be the closure under holomorphic functional calculus of $\mathbb{C}\Gamma \otimes \mathscr{R}$ in $C^*_r(\Gamma) \otimes \mathscr{K}$.*

(i) *If $A \in \mathscr{C}$, $A = (a_{ij})$ with $a_{ij} \in C^*_r(\Gamma)$, then*

$$N_k(A) = \left(\sum_{i,j \in \mathbb{N}} \nu_k(a_{ij})^2 \right)^{1/2} < \infty, \qquad \forall k \in \mathbb{N}$$

where

$$\nu_k(a) = \left(\sum_{g \in \Gamma} (1 + |g|)^{2k} |a(g)|^2 \right)^{1/2}, \ \forall a \in C^*_r(\Gamma);$$

($C^*_r(\Gamma)$ is regarded as a subspace of $l^2(\Gamma)$ in the usual way: $a \in C^*_r(\Gamma)$ is identified with the function $a(\delta_e) \in l^2(\Gamma)$).

(ii) *Let $\tau \in Z^p_\lambda(\mathbb{C}\Gamma)$ be a cyclic p-cocycle, with $p \geq 1$, which is continuous with respect to the norm ν_k for some $k \in \mathbb{N}$. Then $\tau \# tr \in Z^p_\lambda(\mathbb{C}\Gamma \otimes \mathscr{R})$ is continuous with respect to N_k and extends to $\mathscr{C}$.*

Proof. (i) We shall construct a subalgebra $\mathscr{B}$ of $C^*_r(\Gamma) \otimes \mathscr{K}$, which contains $\mathbb{C}\Gamma \otimes \mathscr{R}$, is closed under holomorphic functional calculus (see [8], Chap. I, Appendix 3) and such that $N_k(A) < \infty$ for any $A \in \mathscr{B}$ and $k \in \mathbb{N}$.

To this end, we let Δ, resp. D, be the (unbounded) operator in $l^2(\mathbb{N})$, resp. $l^2(\Gamma)$, defined by:

$$\Delta\delta_j = j\delta_j \quad (j \in \mathbb{N}), \quad \text{resp.} \quad D\delta_g = |g|\delta_g \quad (g \in \Gamma).$$

Next, we consider the (unbounded) derivations $\partial = ad D$ of $\mathscr{L}(l^2(\Gamma))$ and $\tilde{\partial} = ad(D \otimes I)$ of

$\mathscr{L}(l^2(\Gamma)\otimes l^2(\mathbb{N}))$, and set:

$$\mathscr{B} = \{A \in C_r^*(\Gamma)\otimes\mathscr{K} \mid \bar{\partial}^k(A)\circ(I\otimes\Delta) \text{ is bounded } \forall k\in\mathbb{N}\}.$$

First we claim that $\mathscr{B}$ contains $\mathbb{C}\Gamma\otimes\mathscr{R}$. Indeed, if $A = \lambda(g)\otimes S$, with $g\in\Gamma$, λ the left regular representation and $S\in\mathscr{R}$, then $\bar{\partial}^k(A)\circ(I\otimes\Delta) = \partial^k(\lambda(g))\otimes S\Delta$; but both $\partial^k(\lambda(g))$ and $S\Delta$ are bounded.

Secondly, since $\{T\in C_r^*(\Gamma)\otimes\mathscr{K} \mid T\circ(I\otimes\Delta) \text{ bounded}\}$ is evidently a left ideal, $\mathscr{B}$ is a subalgebra, in fact a left ideal in

$$\mathscr{B}_\infty = \bigcap_{k\geq 0} \text{Domain } \bar{\partial}^k.$$

Now $\mathscr{B}_\infty$ is stable under holomorphic functional calculus and, therefore, so are its left ideals.

Finally, let us check that $N_k(A) < \infty$, $\forall A\in\mathscr{B}$. As $D\delta_e = 0$, one has

$$(\bar{\partial}^k(A)\circ I\otimes\Delta)(\delta_e\otimes\delta_j) = j\sum_{i\in\mathbb{N}} \bar{\partial}^k a_{ij}(\delta_e)\otimes\delta_i$$

$$= j\sum_{(g,i)\in\Gamma\times\mathbb{N}} |g|^k a_{ij}(g)\delta_g\otimes\delta_i.$$

Thus, there exists $C<\infty$ such that

$$\sum_{(g,i)\in\Gamma\times\mathbb{N}} |g|^{2k}|a_{ij}(g)|^2 < Cj^{-2};$$

therefore,

$$\sum_{i,j\in\mathbb{N}}\sum_{g\in\Gamma} |g|^{2k}|a_{ij}(g)|^2 < \infty, \qquad \forall k\in\mathbb{N},$$

which implies $N_k(A) < \infty$, $\forall k\in\mathbb{N}$.

(ii) We need to show that, $\forall A^0,\ldots,A^p\in\mathbb{C}\Gamma\otimes\mathscr{R}$,

$$|(\tau_c \# tr)(A^0,\ldots,A^p)| \leq const\cdot N_k(A^0)\ldots N_k(A^p).$$

But

$$(\tau_c \# tr)(A^0,\ldots,A^p) = \sum_{i_0,i_1,\ldots,i_p\in\mathbb{N}} \tau_c(a^0_{i_0,i_1}\ldots a^p_{i_p,i_0})$$

and, since by hypothesis

$$|\tau_c(a^0_{i_0,i_1},\ldots,a^p_{i_p,i_0})| \leq const\cdot\nu_k(a^0_{i_0,i_1})\ldots\nu_k(a^p_{i_p,i_0})),$$

the required estimate follows from the inequality

$$\sum_{i_0,i_1,\ldots,i_p\in\mathbb{N}} \alpha^0_{i_0,i_1}\ldots\alpha^p_{i_p,i_0} \leq \prod_{l=0}^p\left(\sum_{i,j\in\mathbb{N}}(\alpha^l_{ij})^2\right)^{1/2}, \qquad \forall\alpha^l_{ij}\geq 0,$$

valid for $p\geq 1$. □

We shall now apply the previous results to prove that Gromov's hyperbolic groups satisfy the Novikov conjecture. For technical reasons, it will be convenient to work with a larger class of finitely generated groups, namely the groups Γ satisfying the following two conditions:

(PC) (Polynomial Cohomology) *for any* $\xi\in H^*(\Gamma,\mathbb{C})$, *there exists a representative* $c\in Z_\alpha^*(\Gamma:\Gamma)$, $[c]=\xi$, *which has polynomial growth (with respect to some word length function* $g\mapsto|g|$, $g\in\Gamma$*)*;

(RD) (Rapid Decay) *there exists* $k \in \mathbb{N}$ *and* $C > 0$ *such that*

$$\|a\|^2_{C^*_r(\Gamma)} \leq C \sum_{g \in \Gamma} (1 + |g|)^{2k} |a(g)|^2, \qquad \forall a \in \mathbb{C}\Gamma.$$

A finitely generated group Γ which is hyperbolic (with respect to some word metric) enjoys both these properties. Indeed, according to a result of Gromov (c.f. [17], 8.3T), any $\xi \in H^p(\Gamma, \mathbb{C})$, with $p \neq 1$, can be represented by a bounded cocycle. On the other hand, the inequality $\|a\| \leq const \cdot \nu_2(a)$, $\forall a \in \mathbb{C}\Gamma$, has first been proved by U. Haagerup [20] for free groups, then by P. Jolissaint [23] for classical hyperbolic groups and recently it was extended by P. de la Harpe [21] to the general hyperbolic groups of Gromov. One advantage of working with the class of groups satisfying (PC) and (RD) is that it is closed under the direct product operation.

(6.5) Proposition. *Let* Γ *be a finitely generated group satisfying (PC) and (RD). Then* $H^*_{ex}(\Gamma, \mathbb{C}) = H^*(\Gamma, \mathbb{C})$.

Proof. Let $c \in Z^{2q}_\alpha(\Gamma : \Gamma)$ be a cocycle of polynomial growth. This means that there exist a constant $B > 0$ (depending on c) and an integer $m \geq 0$ such that

$$|c(1, g_1, g_1 g_2, \ldots, g_1 \ldots g_p)| \leq B(1 + |g_1|)^{2m} \ldots (1 + |g_p|)^{2m},$$

for all $g_1, \ldots, g_p \in \Gamma$, where $p = 2q$.

The following idea for estimating $\tau_c \in Z^p_\lambda(\mathbb{C}\Gamma)$ is due to Jolissaint [23]. Let $f^0, \ldots, f^p \in \mathbb{C}\Gamma$ and denote $\varphi^0 = |f^0|$, $\varphi^j = |f^j|(1 + |\cdot|)^{2m}$, $j = 1, \ldots, p$. Then

$$\begin{aligned} |\tau_c(f^0, \ldots, f^p)| &\leq B^p \sum_{g_0 g_1 \ldots g_p = 1} \varphi^0(g_0)\varphi^1(g_1) \ldots \varphi^p(g_p) \\ &= B^p(\varphi^1 * \cdots * \varphi^p * \varphi^0)(1) \leq B^p \|\varphi^1 * \cdots * \varphi^p * \varphi^0\|_2. \end{aligned}$$

Using now (RD), one obtains

$$\begin{aligned} |\tau_c(f^0, \ldots, f^p)| &\leq B^p C^q \nu_k(\varphi^1) \ldots \nu_k(\varphi^p) \|\varphi^0\|_2 \\ &= B^p C^p \nu_0(f^0) \nu_{k+m}(f^1) \ldots \nu_{k+m}(f^p). \end{aligned}$$

According to Lemma (6.4) (ii), $\tau_c \# tr$ extends continuously to $\mathscr{C}$. Since the inclusion $\mathscr{C} \subsetneq C^*_r(\Gamma) \otimes \mathscr{K}$ induces an identification of $K_0(\mathscr{C})$ with $K_0(C^*_r(\Gamma))$ (cf. [8; Part I, Appendix 3]), it follows that $[c] \in H^*_{ex}(\Gamma, \mathbb{C})$. □

(6.6) Theorem. *Let* Γ *be a finitely generated group satisfying (PC) and (RD),* M *a compact, oriented, smooth manifold, and* $\psi: M \to B\Gamma$ *a continuous map. Then all higher signatures* $Sign_\xi(M, \psi)$, $\xi \in H^*(\Gamma, \mathbb{C})$ *are oriented homotopy invariants.*

Proof. For M even-dimensional, the theorem follows from Propositions (6.3) and (6.5). If M is odd-dimensional, one replaces $\psi: M \to B\Gamma$ by $\psi \times I: M \times S^1 \to B\Gamma \times S^1 = B(\Gamma \times \mathbb{Z})$. □

In the even-dimensional case, we can formulate a slightly more precise result, which extends the classical Hirzebruch Signature Theorem to higher Γ-cocycles.

(6.7) Theorem. *With the above notation, assume in addition that* M *has even dimension* $2n$. *Let* $\xi = [c] \in H^{2q}(\Gamma, \mathbb{C})$ *with* $c \in Z^{2q}_\alpha(\Gamma : \Gamma)$ *of polynomial growth. Choose a representative* $H = H^* \in M_k(\mathbb{C}\Gamma)$ *for the Mishchenko symmetric signature* $\sigma[M, \psi] \in L^0(\mathbb{C}\Gamma)$ *and let*

$Spec H \subset \mathbb{R} - \{0\}$ be its spectrum in the regular representation on $l^2(\Gamma)$. Then the function of several complex variables

$$F_c(\lambda_0, \ldots, \lambda_{2q}) = \tau_c((\lambda_0 - H)^{-1}, \ldots, (\lambda_{2q} - H)^{-1}),$$

initially defined on a sufficiently small polydisc around the origin, admits an analyic extension to $(\mathbb{C} - Spec H)^{2q+1}$. In particular, if

$$e^{\pm} = \frac{1}{2\pi i}\int_{C^{\pm}} (\lambda - H)^{-1} d\lambda \quad (C^{\pm} = \text{contour around } \mathbb{R}^{\pm} \cap Spec H)$$

are the projections corresponding to the positive, resp. negative, part of $H \in \mathscr{L}(l^2(\Gamma) \otimes \mathbb{C}^k)$, then one can define, by analytic continuation,

$$\tau_c(e^{\pm}, \ldots, e^{\pm}) = \frac{1}{(2\pi i)^{2q+1}} \int_{(C^{\pm})^{2q+1}} F_c(\lambda_0, \ldots, \lambda_{2q}) d\lambda_0 \ldots d\lambda_{2q}.$$

The difference

$$Sign_c(H) = \tau_c(e^+, \ldots, e^+) - \tau_c(e^-, \ldots, e^-)$$

is an oriented homotopy invariant of M. Moreover, one has

$$Sign_c(H) = \frac{2^n}{(2\pi i)^q} \frac{q!}{(2q)!} \langle \mathscr{L}(M)\psi^*(\iota(\xi)), [M] \rangle.$$

Proof. To define F_c for $|\lambda_j|$ small (as well as for $|\lambda_j|$ sufficiently large) one uses the polynomial growth of $c \in Z_\alpha^{2q}(\Gamma{:}\Gamma)$. The fact that F_c extends analytically to $(\mathbb{C} - Spec H)^{2q+1}$ is a consequence of Proposition (6.5) (or rather of its proof). Notice that $Spec H$ remains the same when H is viewed in $M_k(C_r^*(\Gamma))$, so that $e^{\pm}$ can also be regarded as elements of $M_k(C_r^*(\Gamma))$. As such,

$$[e^+] - [e^-] = j_L(\sigma[M, \psi]) \in K_0(C_r^*(\Gamma)),$$

by the very definition of the map j_L. It follows from (6.2) that

$$[e^+] - [e^-] = j_K(ind_\Gamma \tilde{D}^{\psi}),$$

therefore,

$$\begin{aligned} Sign_c H &= \tilde{\tau}_c(e^+, \ldots, e^+) - \tilde{\tau}_c(e^-, \ldots, e^-) \\ &= [\tau_c](ind_\Gamma \tilde{D}_\psi) = Ind_{(c,\Gamma)} \tilde{D}_\psi. \end{aligned}$$

□

Although we have taken a more direct route, our method does imply, for hyperbolic groups, Kasparov's strengthened version of the Novikov conjecture (cf. [24], p. 192):

($\mathrm{SNC}_{\beta \otimes \mathbb{Q}}$) *The homomorphism $\beta: RK_*(B\Gamma) \to K_*(C_r^*(\Gamma))$ is rationally injective.*

(6.8) THEOREM. *Kasparov's Strong Novikov Conjecture ($SNC_{\beta \otimes \mathbb{Q}}$) holds for any finitely generated group Γ satisfying (PC) and (RD).*

Proof. Since replacing Γ by $\Gamma \times \mathbb{Z}$ switches the parity of the K-groups, it suffices to treat the even case. We recall that $B\Gamma$ can be realized as a locally compact, σ-compact space, and that $RK_0(B\Gamma) \overset{def}{=} indlim\{K_0(X) | X \text{ compact}, X \subset B\Gamma\}$. Furtermore, in terms of the Baum–Douglas realization [5] of the K-homology, the map β can be desribed as follows (see also [4]). Let (M, F, ψ) be a topological K_0-cycle on $B\Gamma$, i.e., M is an even-dimensional

$Spin^c$-manifold, F is a vector bundle over M and $\psi: M \to B\Gamma$ a continuous map, and let $D\!\!\!/_F$ be the Dirac operator on M, with twisted coefficients in F. Then

$$\beta(\psi_*[D\!\!\!/_F]) \stackrel{def}{=} ind_{C_r^*(\Gamma)} \mathscr{D}_F,$$

where $\mathscr{D}_F = I \otimes D\!\!\!/_F: C^\infty(M, \mathscr{V} \otimes F \otimes S^+) \to C^\infty(M, \mathscr{V} \otimes F \otimes S^-)$ with $\mathscr{V} = C_r^*(\Gamma) \times_\Gamma \psi^*(E\Gamma)$.

Assume now that $\beta(\psi_*[D\!\!\!/_F]) = 0$, for some virtual bundle F. From (6.1), (6.3), (6.5) and (5.4) it follows that $\langle ch_*(\psi_*[D\!\!\!/_F]), \xi \rangle = 0$, $\forall \xi \in H^*(B\Gamma, \mathbb{Q})$, where $ch_*: K_0(B\Gamma) \to H_{ev}(B\Gamma, \mathbb{Q})$ is the Chern character in K-homology [5]. Since ch_* is a rational isomorphism, the statement follows. □

(6.9) *Remark.* According to a theorem of Gromov–Lawson [19], *a compact manifold which admits a metric with sectional curvature $K \leq 0$ cannot carry a non-flat metric with scalar curvature $\kappa \geq 0$.* Combining Rosenberg's results [30] with our Theorem (6.8) one obtains the following partial extension of the Gromov–Lawson theorem:

No closed $K(\Gamma, 1)$-manifold, with Γ hyperbolic, can admit a metric of positive scalar curvature.

REFERENCES

1. M. F. Atiyah: Elliptic operators, discrete groups and von Neumann algebras. *Asterisque*, **32–33** (1976), 43–72.
2. M. F. Atiyah, R. Bott and V. K. Patodi: On the heat equation and the index theorem. *Invent. math.* **19** (1973), 279–330.
3. M. F. Atiyah and I. M. Singer: The index of elliptic operators I, III. *Ann. Math.* **87** (1968), 484–530 and 546–609.
4. P. Baum and A. Connes: Geometric K theory of Lie groups and foliations. *Preprint IHES*, 1982.
5. P. Baum and R. Douglas: K-theory and index theory. *Operator algebras and applications.* Proc. Symposia Pure Math. **38** (1982) vol. I, pp. 117–173.
6. B. Blackadar: *K-Theory for Operator Algebras.* Mathematical Sciences Research Institute Publications; 5. Springer, New York, (1986).
7. D. Burghelea: The cyclic homology of the group rings. *Comment. Math. Helvetici* **60** (1985), 354–365.
8. A. Connes: Non-commutative differential geometry. *Publ. Math. IHES*, **62** (1985), 41–144.
9. A. Connes: Entire cyclic cohomology of Banach algebras and characters of θ-summable Fredholm modules. *K. theory*, **1** (1988), 519–548.
10. A. Connes and M. Karoubi: Caractère multiplicatif d'un module de Fredholm. *K-Theory*, **2** (1988), 431–463.
11. A. Connes and H. Moscovici: Transgression du caractère de Chern et cohomologie cyclique. *C. R. Acad. Sc. Paris, t. 303, Serie I*, no. 18 (1986), 913–918.
12. A. Connes and H. Moscovici: Conjecture de Novikov et groupes hyperboliques. *C. R. Acad. Sc. Paris.* (in press).
13. H. Donnelly: Spectrum and the fixed point set of isometries. I. *Math. Ann.* **224** (1976), 161–170.
14. E. Getzler: Pseudodifferential operators on supermanifolds and the Atiyah-Singer index theorem. *Commun. Math. Phys.* **92** (1983), 163–178.
15. E. Getzler: A short proof of the local Atiyah-Singer index theorem. *Topology*, **25** (1986), 111–117.
16. P. B. Gilkey: *Invariance theory, the heat equation, and the Atiyah-Singer index theorem.* Publish or Perish, Wilmington (1984).
17. M. Gromov: Volume and bounded cohomology. *Publ. Math. IHES*, **56** (1983), 213–307.
18. M. Gromov: Hyperbolic groups. In *Essays in group theory*, S. M. Gersten, (Editor); Mathematical Sciences Research Institute Publication; 8. 75–264, Springer, New York, (1987).
19. M. Gromov and H. B. Lawson: Positive scalar curvature and the Dirac operator on complete Riemannian manifolds. *Publ. Math. IHES*, **58** (1983), 83–196.
20. U. Haagerup: An example of non nuclear C^*-algebra which has the metric approximation property. *Invent. math.* **50** (1979), 279–293.
21. P. de la Harpe: Sur les algèbres d'un groupe hyperbolique. A paraitre au *C. R. Acad. Sci.*
22. D. Husemoller: *Fibre bundles.* McGraw-Hill, New York (1966).
23. P. Jolissaint: Les Fonctions à décroissance rapide dans les C^*-algebres réduites de groupes. *Thèse*, Université de Genève (1987).

24. G. G. KASPAROV: Equivariant KK-theory and the Novikov conjecture. *Invent. math.* **91** (1988), 147–201.
25. G. G. KASPAROV: K theory, group C^* algebras and higher signatures. Conspectus parts I, II, Preprint Chernogolovska (1981).
26. A. S. MISHCHENKO: Homotopy invariants of non simply connected manifolds. Rational invariants I. *Izv. Akad. Nauk. SSSR.* **34** (1970) n° 3, 501–514.
27. A. S. MISHCHENKO and A. T. FOMENKO: The index of elliptic operators over C^*-algebras. *Izv. Akad. Nauk SSSR, Ser. Mat.* **43** (1979), 831–859 = *Math. U.S.S.R.-Izv.* **15** (1980), 87–112.
28. S. P. NOVIKOV: Pontrjagin classes, the fundamental group and some problems of stable algebras. In *Essays on Topology and Related Topics*, Mémoires dédiés à G. de Rham, Springer, Berlin (1969), 147–155.
29. A. RANICKI: The algebraic theory of surgery, I, II. *Proc. London Math. Soc.* (3) **40** (1980), 87–192, 193–283.
30. J. ROSENBERG: C^*-algebras, positive scalar curvature and the Novikov conjecture. *Publ. Math. IHES*, **58** (1983), 409–424.
31. I. M. SINGER: Some remarks on operator theory and index theory. *Lecture Notes in Math.*, n° 575 (1977), 128–138.
32. E. H. SPANIER: *Algebraic Topology*. Springer, New York (1966).
33. H. WIDOM: Families of pseudodifferential operators. In *Topics in Functional Analysis*, I. Gohberg, M. Kac (Editors). pp. 345–395, Academic Press (1978).
34. H. WIDOM: A complete symbolic calculus for pseudodifferential operators. *Bull. Sc. Math.* **104** (1980), 19–63.
35. A. WASSERMAN: Preprint, Berkeley, (1987).

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